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XC. *Some Properties of Valve Oscillators with Coupled Circuits.* *By* R. R. NIMMO, *Ph.D., Lecturer in Physics, University of Western Australia* *.

A NUMBER of workers have discussed the properties of a pair of magnetically coupled circuits, in series with one of which is an oscillating arc, and the behaviour of this circuit is well known †. Much less attention has been given to the theory of valve oscillators with coupled circuits, but at least two such circuits have been worked out ‡. These cases suggest that the behaviour of the valve oscillator with a coupled secondary is likely to be very similar to that of an arc-maintained coupled circuit of the type already analysed, and it seems that it is often assumed that the theory of the arc oscillator is directly applicable to coupled circuits in which oscillations are excited by means of valves.

Such use of the arc oscillator theory is not justified, and, as is shown below, valve circuits of simple types may occasionally behave in a manner which is quite different from that of a similar circuit maintained by

* Communicated by Lord Rutherford, O.M., D.Sc., F.R.S.

† See, for example, Palmer, 'Wireless Principles and Practice' (1928); or Lee and Gill, Journ. I. E. E. vol. lxiii. p. 703 (1925).

‡ B. van der Pol, Phil. Mag. vol. xliii. p. 700 (1922); and A. Andronov and A. Witt, Tech. Phys. of the U.S.S.R. vol. i. p. 249 (1934).

an arc. This difference is not brought about by any difference in the inherent properties of arcs and of valves, but because in practice the arc is placed in series with the oscillating circuit, whereas the valve is used in parallel with it. In this paper the term "arc-maintained circuit" is restricted to the case in which an arc is in series with an oscillating circuit, while the term "valve-maintained circuit" refers only to circuits in which the valve is in parallel with the oscillating circuit.

It is known, for example, that the amplitude of oscillation of an arc circuit is decreased when it is made to supply energy to a coupled secondary circuit, but in the case of suitable valve circuits the amplitude of oscillation may be increased when a secondary circuit is coupled to the oscillator.

It is also known that on cyclically altering the mutual inductance between a secondary circuit and a valve oscillator repeated discontinuous frequency changes may occur *. As an investigation of the theory of the arc oscillator shows that this phenomenon could not occur with such an oscillator the circumstances in which it does occur may be determined only by investigating the equations appropriate to a valve oscillator.

In giving a treatment of valve oscillators it is essential to remember that the valve has non-linear characteristics. To take the dynatron as an example, we know that its resistance is not constant, and we therefore suppose that the current † through it is some function of v, the potential across it; this function we write as $f(v)$. On obtaining a differential equation for the circuit being considered this function and its derivatives with respect to time are found in the equation. As an accurate solution is not possible we confine ourselves to the case in which v is very nearly a sinusoidal function of the time. Now as both v and $f(v)$ are periodic functions of the time, and each has the same period, which we may call $2\pi/\omega$, we imagine v and $f(v)$ to be expressed as Fourier series, and consider only the components of angular frequency ω. If $f(v)$ can be expressed as a series of powers of v, and if,

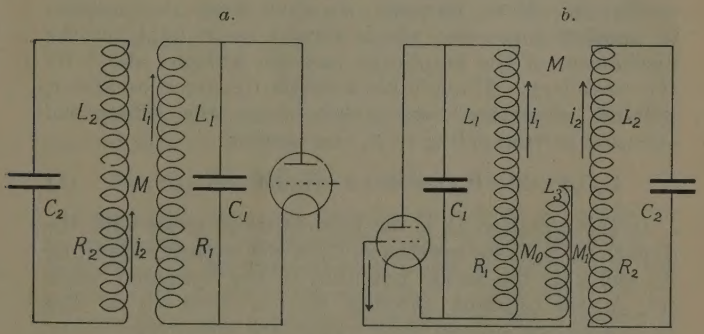
* Clinker and Kinman, 'Wireless Engineer and Experimental Wireless,' vol. ix. p. 11 (1932).

† In all cases in which currents and voltages are mentioned it is to be understood that only the fluctuating components of the actual currents and voltages are being considered.

in addition, v differs very little from $a \sin \omega t$, the approximate value of $f(v)$ is taken as $a\phi(a) \sin \omega t$, where $\phi(a)$ can be expressed as a series of powers of a in which the coefficients are simply related to those occurring in the power series for $f(v)$. In order then to determine from the differential equation the properties of these nearly sinusoidal oscillations we substitute $a \sin \omega t$ for v and $a\phi(a) \sin \omega t$ for $f(v)$, and equate to zero the coefficients of $\sin \omega t$ and $\cos \omega t$. The resulting equations contain the quantity $\phi(a)$, which is regarded as a constant for any one amplitude of oscillation.

Let us now examine the properties of the dynatron and back-coupled valve oscillators shown in fig. 1. In the case

Fig. 1.



of the dynatron (fig. 1 (*a*)) we suppose that the current through it is given by $f(v)$, some function of v, the voltage applied to its anode. In the case of the back-coupled valve we assume that the anode current is given by $f(v_p + \mu v_g)$, where v_p and v_g are the values of the anode and grid voltages, and μ is the amplification factor of the valve.

On referring to fig. 1 (*b*) it is seen that in the case of the particular type of valve oscillator considered there are three mutual inductances of which account is taken. These are the mutual inductance M between the primary and secondary coils L_1 and L_2, the mutual inductance M_0 between the grid and anode coils L_3 and L_1, and the mutual inductance M_1 between the grid and secondary coils L_3 and L_2. No account is taken of any small current

flowing in the grid coil L_3, but in order to enable signs to be given to the mutual inductances M_0 and M_1 it is necessary to choose some convenient positive direction for this current. The direction chosen is that indicated by the arrow on the left-hand side of fig. 1 (*b*).

In an appendix is given a differential equation which may be used for either valve oscillator; from this equation it is possible to obtain the properties of the dynatron and back-coupled valve oscillators.

We wish now to determine the circumstances in which the amplitude of oscillation of the valve circuit is greater when the secondary is present than when it is absent, and, as is shown in outline in the appendix, the necessary analysis may be carried out from the approximate solution of the differential equation appropriate to the valve oscillators. Here, however, we shall treat the problem in another way—one which throws some light on the mechanism of the amplitude increase brought about by the secondary. If one takes a simple dynatron oscillator, without secondary, and writes down the differential equation corresponding to it, one obtains

$$L_1C_1d^2v/dt^2+R_1C_1dv/dt+v+(L_1d/dt+R_1)f(v)=0. \quad (1)$$

In this equation L_1 is the inductance in series with the dynatron, R_1 is its resistance, C_1 is the capacity shunting it, and v is the anode potential of the dynatron, while $f(v)$ is the current through it. If $v=a \sin \omega t$ and $f(v)=a\phi(a) \sin \omega t$, and these values are substituted in (1), we have, on equating to zero the coefficients of $\cos \omega t$,

$$\phi(a)+R_1C_1/L_1=0; \quad . \quad . \quad . \quad . \quad (2)$$

$\phi(a)$ is the conductance of the dynatron. Eq. (2) means that if oscillations are possible at all, *i. e.*, if a is not zero, the amplitude of oscillation is always such that (2) is satisfied. If R_1 is decreased we should expect the amplitude of oscillation to increase. From eq. (2) we see that $\phi(a)$ is also increased—that is, $d\phi(a)/da$ is positive.

Now consider the effect of the presence of the secondary. As in the ordinary transformer the current in the secondary circuit changes the effective values of the primary resistance and the primary inductance. If R_1' is the effective value of the primary resistance, and L_1' the effective

value of the primary inductance when the secondary is in a given position, we have

$$R_1' = R_1 + R_2\,\omega^2M^2/Z_2^2$$

and $$L_1' = L_1 - \omega M^2(\omega L_2 - 1/\omega C_2)/Z_2^2,$$

where M is the mutual inductance between primary and secondary circuits, R_2, L_2, and C_2 are the resistance, inductance, and capacity in the secondary circuit, and $Z_2^2 = R_2^2 + (\omega L_2 - 1/\omega C_2)^2$.

When the secondary is in place eq. (2) becomes

$$\phi(a) + R_1'C_1'/L_1' = 0. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Clearly the secondary will increase the amplitude of oscillation only if

$$R_1'C_1'\,L_1' < R_1C_1/L_1. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Now R_1' is always greater than R_1, so that to satisfy (4) L_1' must also be greater than L_1, and in addition

$$L_1'/L_1 > R_1'/R_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

On substituting the values given above for L_1' and R_1' we find that (5) reduces to

$$L_1R_2/L_2R_1 < 1/\omega^2L_2C_2 - 1. \quad . \quad . \quad . \quad . \quad (6)$$

Expression (6) shows that the amplitude of oscillation is increased by the secondary only if the frequency of oscillation of the circuit as a whole is lower than the resonant frequency of the secondary. This will normally require that the resonant frequency of the primary is lower than that of the secondary. This expression (6) also suggests that the secondary will cause an increase in the amplitude of oscillation if only C_2 is made sufficiently small. Nevertheless, though this is true, the increase will be of negligible magnitude when the primary and secondary circuits are tuned to very different frequencies, for when this is the case the current flowing in the secondary circuit is small and it has little effect upon the oscillations of the primary. For this reason, as (6) suggests, L_1R_2/L_2R_1 should be made small if it is desired to have considerable increases in amplitude of oscillation when the primary and secondary circuits are nearly in tune with one another. Unfortunately this expression (6) contains the unknown ω, and uncertainties in its value make further analysis impossible.

Experimentally one finds that ω falls when the value of C_2 is changed from small to larger values. It has been mentioned that if the amplitude of oscillation is increased by the secondary the frequency generated must be lower than the resonant frequency of the secondary; but in the region with which we are concerned the frequency generated is always lower than the resonant frequency of the primary, and it may thus be possible that the secondary increases the amplitude of oscillation even when it is tuned to a lower frequency than is the primary. That this is sometimes the case is shown rigorously in the appendix.

The description of the phenomenon is then as follows. When the frequency of oscillation of the circuit is lower than the resonant frequency of the secondary the secondary circuit will take a current which is nearly 90° ahead of the E.M.F. inducing it. This current increases the effective inductance and the effective resistance of the primary circuit. In suitable circumstances, viz., when (6) or (1·9) in the appendix is satisfied, the percentage increase in effective inductance is greater than the percentage increase in effective resistance, and the presence of the secondary reduces $R_1'C_1'/L_1'$ below the value it had when the secondary circuit was absent. Now, as already pointed out, any reduction in the value of R_1C_1/L_1 results in an increased amplitude of oscillation of the circuit, and in the circumstances described above the presence of the secondary leads to such an increase of amplitude of oscillation.

Let us now consider briefly the more interesting though more difficult case of the back-coupled valve oscillator when coupled to a secondary, as is shown in fig. 1 (*b*). For a simple back-coupled valve oscillator without secondary the differential equation is found to be

$$L_1C_1d^2v/dt^2+R_1C_1dv/dt+v+(L_1-\mu M_0)df(v)/dt+R_1f(v)=0.$$

Here M_0 is the mutual inductance between grid and anode coils, and v is written for $v_p+\mu v_g$. On substituting $v=a\sin\omega t$ and $f(v)=a\phi(a)\sin\omega t$, and equating to zero the coefficients of $\cos\omega t$, we have

$$\phi(a)+R_1C_1/(L_1-\mu M_0)=0. \quad . \quad . \quad . \quad . \quad (7)$$

Again, if oscillations are possible at all the amplitude adjusts itself so that (7) is satisfied. $\phi(a)$, the anode

conductance of the valve, is a positive quantity; for oscillations to be possible $L_1-\mu M_0$ is negative. As with the dynatron, the amplitude of oscillation increases when R_1 is decreased, and from (7) we see that an increase of amplitude results when $R_1C_1/(\mu M_0-L_1)$ is decreased.

When a secondary circuit is present there are changes in the effective values of R_1, L_1, and M_0. The secondary will increase the amplitude of oscillation of the circuit if

$$R_1C_1/(\mu M_0-L_1)>R_1'C_1/(\mu M_0'-L_1'), \quad . \quad . \quad (8)$$

where R_1' and L_1' have the same meanings as in the case of the dynatron and M_0' is the effective value of M_0 in the presence of the secondary. We know R_1' and L_1' in terms of the frequency and the circuit constants, but the value of M_0' is not immediately obvious.

When unit current flows in the coil L_1 the E.M.F. in the grid coil is ωM_0, and this E.M.F. is in phase with the inductive E.M.F. ωL_1 in the anode coil. In the secondary coil there is an E.M.F. ωM, which gives a current of magnitude $\omega M/Z_2$ in the secondary; this current induces an E.M.F. ω^2MM_1/Z_2 in the grid coil, where M_1 is the mutual inductance between secondary and grid coil and Z_2 is as defined above. Of this E.M.F. in the grid coil the component in phase with the E.M.F. induced by the primary coil is $-\omega^2MM_1(\omega L_2-1/\omega C_2)/Z_2^2$. When no secondary is present the E.M.F. in the grid coil is ωM_0; with a secondary the E.M.F. of the same phase in the grid coil is $\omega M_0-\omega^2MM_1(\omega L_2-1/\omega C_2)/Z_2^2$.

Thus

$$M_0'=M_0-\omega MM_1(\omega L_2-1/\omega C_2)/Z_2^2. \quad . \quad . \quad (9)$$

On substituting the values of R_1', L_1', and M_0' in (8) we find that the secondary increases the amplitude of oscillation of the primary when

$$R_1(1-\mu M_1/M)(\omega L_2-1/\omega C_2)>\omega R_2(\mu M_0-L_1). \quad . \quad (10)$$

In the appendix it is shown that $\mu M_1/M$ is normally a positive quantity, which is usually greater than unity. When this is the case the amplitude of oscillation of the circuit can be increased only if $\omega L_2-1/\omega C_2$ is negative—that is, if the frequency of oscillation of the circuit as a whole is lower than the resonant frequency of the secondary circuit. If, however, $1-\mu M_1/M$ is positive, and this can be arranged by using special circuits, condition (10)

is satisfied only if $\omega L_2 - 1/\omega C_2$ is positive, so that for such special circuits the amplitude of oscillation will be increased only if the resonant frequency of the secondary is lower than the frequency of oscillation of the circuit as a whole. Normally, however, when $1-\mu M_1/M$ is negative one finds that the conditions in which the secondary increases the amplitude of oscillation of the circuit for a back-coupled valve oscillator are similar to those required in the case of the dynatron oscillator.

Nevertheless the mechanism in the case of the back-coupled valve is different from that occurring for the dynatron. If the secondary is tuned to a frequency higher than the frequency of oscillation of the circuit the presence of the secondary brings about increases in the effective primary resistance, the effective primary inductance, and the effective mutual inductance between the anode and grid coils. If $\mu M_1/M > 1$ the change in the effective value of μM_0 is greater than that in the self inductance L_1, and as a result of the presence of the secondary there are increases in the effective values of both the numerator and denominator of the expression $R_1C_1/(\mu M_0 - L_1)$. In suitable circumstances, viz., when (10) or (1.8) in the appendix is satisfied, the percentage increase in the denominator is greater than that in the numerator, and the consequent decrease in $R_1'C_1'/(\mu M_0' - L_1')$ brings about an increase in the amplitude of oscillation.

One very special case of the back-coupled valve oscillator should be mentioned here, though the theory is too complicated to give at this stage. If

$$\mu M_0 - L_1 > 0,\ M - \mu M_1 > 0,\ R_1 M(M - \mu M_1) - R_2 L_1(\mu M_0 - L_1) > 0, \text{ and } a \text{ is not zero } *, \quad . \quad (11)$$

and these conditions are difficult to satisfy simultaneously, it may be shown that the presence of the secondary always increases the amplitude of oscillation of the primary, whatever the value of the secondary capacity. As $M - \mu M_1$ is positive we see from (10) that the frequency of oscillation of the circuit must be higher than the resonant frequency of the secondary. It may also be shown that for any given setting of the secondary

* This requires that there is available a valve which will produce oscillations in such a circuit.

capacity, when the circuit constants are such that (11) is satisfied, the coupled circuits have only one possible frequency of oscillation. For this arrangement then, in which the primary and secondary must be tightly coupled to one another, the frequency of oscillation is always higher than the resonant frequency of the secondary, and, in addition, the circuit does not exhibit the phenomenon known as ziehen ; furthermore, whatever the secondary capacity the amplitude of oscillation is greater than if the secondary circuit is removed.

In view of what has been said already it should be clear that the increase in amplitude of oscillation brought about by the secondary is a necessary consequence of the conditions for oscillation of the particular oscillator discussed. With other types of oscillator, where the conditions for oscillation are not the same as those considered above, the secondary circuit may never increase the amplitude of oscillation, as happens, for example, in the case of the ordinary arc oscillator.

Let us now investigate the effect of cyclic alterations in the mutual inductance between the primary coil L_1 and the secondary coil L_2. In order to understand why such alterations produce discontinuous frequency changes one must have information as to the stability of the oscillations produced by the circuit.

It may be shown, both theoretically and experimentally, that for certain values of the circuit constants there are two possible stable modes of oscillation of the circuit. The frequencies generated in the two cases are not the same, and in general the corresponding amplitudes of oscillation are also different. Of the two modes of oscillation that having the larger amplitude of oscillation is the more stable ; indeed, when this mode is generated any infinitesimal change in the constants of the circuit cannot lead to the replacement of the oscillation of large amplitude by the one of small amplitude. It is otherwise with the mode of oscillation which has the smaller amplitude ; this mode is stable only as long as its amplitude is not too small ; but if by some alteration of the circuit constants the amplitude of oscillation is sufficiently reduced this mode of oscillation is replaced by the other mode to which corresponds the greater amplitude.

Now although it is easy to obtain an expression which gives a relation between the circuit constants at which

this change in mode of oscillation is on the point of taking place, the expression obtained is too complicated to be of use. We must content ourselves with the knowledge that as long as the existing mode of oscillation has an amplitude greater than that of the other possible mode the existing mode is in no immediate danger of being replaced; but when, by some alteration of the circuit constants, the existing mode of oscillation is made to have a smaller amplitude than that of the other mode, it is in serious danger of being replaced, and if only the alteration of circuit constants can be carried far enough the existing mode will be replaced by the other mode of oscillation, the frequency of oscillation will change discontinuously, and the amplitude of oscillation will increase, perhaps considerably. If, now, the alteration of circuit constants is reversed in direction, a point will be reached where the existing mode has an amplitude smaller than that of the original mode of oscillation, and if the circuit constants can be altered sufficiently another discontinuous frequency change takes place, and the circuit then oscillates in the mode which was first generated. As is well known from experiments this behaviour, often referred to as the ziehen phenomenon, is exhibited by a coupled-valve or arc-oscillator circuit in which there is sufficient coupling between primary and secondary circuits when the secondary capacity is suitably altered.

One wishes, however, to determine theoretically whether a similar phenomenon will occur when some other circuit parameter is suitably altered. This can readily be done if the essential part of the process is realized. Suppose that the possible modes of oscillation are called I. and II. and suppose that with the given circuit constants the amplitude of mode I. is greater than that of mode II. If the circuit is oscillating in mode I., the mode with the greater amplitude, we will say that the oscillation produced is absolutely stable, whereas if it is oscillating with a small amplitude in mode II. we will call the oscillation conditionally stable.

Suppose that the absolutely stable oscillation is being generated. Now let the circuit constants be altered. The amplitudes of oscillation corresponding to modes I. and II. will of course be altered, but mode I. will continue to be absolutely stable unless the circuit constants pass through a value for which the amplitudes of modes I. and

II. are equal. If this happens, and only if this happens, the amplitude of mode I. is reduced below that of modeII., and mode I., instead of being absolutely stable, becomes conditionally stable. Further alteration of the circuit constants may produce a discontinuous frequency change. Such discontinuous frequency changes can be produced by the alteration of any circuit constant whatsoever, provided that the nature of the circuit is such that on altering the constant chosen the amplitudes of oscillation in the two possible modes can be made to pass through a condition of equality.

Fortunately it is simple to derive the relation between the circuit constants for which the two modes of oscillation give equal amplitudes *, and from this relationship it is easy to decide whether discontinuous frequency changes are likely to be produced by the alteration of any circuit constant.

In the case of the dynatron it may be shown that the two modes of oscillation have equal amplitudes when

$$(L_1C_1/L_2C_2+L_1R_2/L_2R_1)/(1-M^2/L_1L_2) = (L_1C_1/L_2C_2+R_1R_2C_1/L_2)\ (L_1R_2/L_2R_1+1). \quad (12)$$

We see from equation (12) that the condition for equal amplitudes in the two modes may be satisfied by suitably adjusting the mutual inductance M between the primary and secondary circuits. Suppose that M is much smaller than the value satisfying (12), and that the circuit constants are such that M may be given a value considerably greater than that required to satisfy (12). Now let M be increased. It may be shown that when M is just large enough to allow of two possible modes of oscillation the higher of the two frequencies is the absolutely stable oscillation, and this frequency will be generated with these small values of M. But when M is increased to the value satisfying (12) both modes have the same amplitude. On further increasing M the existing mode becomes conditionally stable, and if M is increased sufficiently this mode is replaced by the absolutely stable lower frequency. If M is now decreased this lower frequency oscillation remains absolutely stable until M has the value satisfying (12), but on further decreasing M the mode becomes conditionally stable, and finally there

* The amplitude to be considered is that of the valve anode current.

is a discontinuous frequency change to the high-frequency oscillation originally produced. This ziehen-like phenomenon is produced on varying the mutual inductance because the circuit is such that M occurs in eq. (12), which gives the condition for equal amplitudes of oscillation in the two modes. The phenomenon can be produced only when M can take values both above and below the value satisfying (12). This condition requires that the higher possible frequency is the absolutely stable one for values of M smaller than that satisfying (12) *. Furthermore, as stated above, the discontinuous-frequency change on increasing M is from a high frequency to a low one.

It is unnecessary to mention in detail the behaviour of a back-coupled valve oscillator with coupled secondary. It should be noted, however, that on changing the mutual inductance between the primary and secondary there is also a change in M_1, the mutual inductance between the secondary and the grid coil. If, however, $\mu MM_1 - M^2$ is a positive quantity which increases with increase of M the phenomena to be expected are similar in every way to those predicted above for the dynatron. Once again the two valve circuits behave similarly, but this similarity of behaviour is understood only if account is taken of the mutual inductance between the secondary and grid coils.

It may be mentioned that for a series arc-maintained coupled circuit the relationship between the circuit constants when both modes of oscillation have the same amplitude is simply $L_1C_1 = L_2C_2$. As this relationship does not contain the mutual inductance no cyclic variation of the mutual inductance between the two circuits can give ziehen-like discontinuous frequency changes.

Experimental Results.

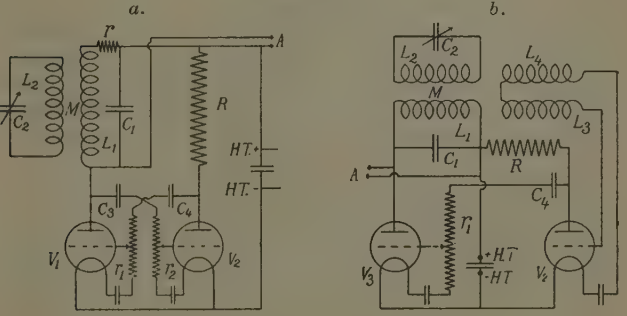
Two special types of valve circuit which were used in testing the theory are shown in fig. 2 (*a*) and (*b*).

The normal frequency of oscillation of these circuits was of the order of 500 cycles per second. At such

* If the resonant frequency of the secondary is greater than that of the primary this condition cannot be satisfied. For M very small it will be satisfied if the resonant frequency of the secondary is slightly less than that of the primary.

frequencies a well-designed resistance-capacity amplifier should produce very small phase errors, and the circuit shown in fig. 2 (*a*) was expected to have properties similar to those possessed by a dynatron oscillator. While a simple dynatron oscillator is to be preferred if a suitable dynatron is available, the circuit shown was found to be very satisfactory and to be capable of producing oscillations even when the dynamic conductance of the L_1C_1 circuit was relatively high ; in this way the circuit used simulated a dynatron with a conductance of high absolute value. It was possible to arrange that the grids of V_1 and V_2 (fig. 2 (*a*)) were never positive, and most measurements were made in conditions in which no grid currents flowed.

Fig. 2.



L_1 = outside margin.

L_1 = 1·063 henry ; resistance 226 ohms. L_2 0·148 henry ; resistance 26·0 ohms. C_1 1/12 μF. M = mutual inductance between L_1 and L_2 ; maximum 0·22 henry. r = variable resistance ; maximum 500 ohms. L_3 and L_4 : large honeycomb coils. C_2 = capacity ; maximum 1·44 μF. r_1 and r_2 : divided megohms. C_3 and C_4 : 0·2 μF. R = 40,000 ohms. V_2: Osram MHL4. V_1 : Philips TB04 with filament underheated. V_3 : American 2A3 valve with an anode resistance stated to be only 765 ohms. Capacities unlettered are blocking condensers. Batteries are not shown.

In order to examine the effect of the secondary on the amplitude of oscillation of the circuit the leads marked A (fig. 2 (*a*)) were connected to a cathode ray oscillograph or to a valve voltmeter. The capacity C_2 was then made zero, and the amplitude of oscillation observed ; C_2 was increased by 0·1 μF steps, and the amplitude noted for

each step. For the lower values of this capacity the amplitude was of the order of 20 per cent. greater than when C_2 was zero, but on increasing the capacity further the amplitude fell again, and in some cases the capacity could be increased sufficiently to bring about the discontinuous frequency change characteristic of this type of circuit. A similar increase in amplitude was observed when C_2 was small, say $0{\cdot}4\ \mu$F, and the mutual inductance between L_1 and L_2 was increased from zero to the maximum possible value ; in this case, however, there was no discontinuous frequency change. The effect of the secondary was also shown in a striking fashion by carrying out an experiment in which the circuit just did not oscillate in the absence of the secondary. If C_2 was about $0{\cdot}4\ \mu$F the circuit burst into oscillation when the secondary was brought up to the primary.

But when the secondary capacity was sufficiently large to cause the circuit to oscillate with a frequency which was higher than that generated by the primary alone, and presumably higher than the resonant frequency of the secondary also, the amplitude of oscillation was smaller than that produced when the secondary was absent.

Experiments were next carried out in order to determine the conditions in which cyclic changes in the mutual inductance were able to produce repeated discontinuous frequency changes. Such discontinuous frequency changes did not occur unless the circuit was arranged so that on increasing the mutual inductance the frequency of oscillation of the circuit was increased. This increase of frequency was associated with a decrease of amplitude of oscillation ; eventually, when the mutual inductance M was sufficiently large, the frequency changed discontinuously to a much lower value, and at the same time the amplitude of oscillation was greatly increased. On decreasing M the frequency rose slightly and the amplitude fell ; later, when a frequency transition occurred, the circuit oscillated with the high frequency which was originally generated with small values of the mutual inductance. An interesting modification of this phenomenon was observed. The circuit was arranged so that the frequency of oscillation was higher than that of the primary alone, but as the filament current of the valve V_1 was less than usual the amplitude of oscillation

was small. On increasing the mutual inductance the amplitude fell and later oscillations ceased; but on increasing the mutual inductance still further there was a sudden appearance of the low-frequency oscillation. On reducing the mutual inductance the amplitude of the low-frequency oscillation fell and the circuit ceased to oscillate; when the mutual inductance was further decreased the high-frequency oscillation was once again generated.

Though these results were in good agreement with the theory given above it was felt that some quantitative measurements were desirable. An attempt was therefore made to verify eq. (12), which, as was pointed out above, gives the relation between the various circuit constants when the dynatron anode current, and hence voltage, has the same value for both oscillation frequencies. The anode voltage has, of course, the same magnitude as the voltage across the coil L_1. In order to see whether the circuit constants satisfied this equation the resistance r and the mutual inductance M (fig. 2 (*a*)) * were given definite values, and the capacity C_2 was adjusted so that the amplitude of oscillation of the circuit, as read on a valve voltmeter, was the same for both frequencies of oscillation. When this occurred the circuit constants were expected to have values which satisfied eq. (12). In this equation C_2 was regarded as the unknown, and as all the other quantities were known, C_2 was calculated in terms of them. This calculated value of C_2 was then compared with the actual value of C_2 for which the amplitudes of the two frequencies were the same. The results are given in Table I. In determining C_2 from (12) the values used for the fixed constants of the circuit were taken from those given in the list under fig. 2.

It is not necessary to give details of the behaviour of the simple back-coupled valve oscillator. The phenomena observed with a valve circuit similar to that shown in fig. 1 (*b*) did not differ in any important way from those observed with the dynatron circuit. In this case, however, no attempt was made to verify an equation similar to (12) giving the condition that the oscillating valve

* The circuit used was slightly different from that shown in fig. 2 (*a*). The valve V_2 was a Philips E 406 and R was 12,000 ohms; the valve voltmeter was connected across portion of the resistance r_2.

anode currents were equal in amplitude for both frequencies of oscillation.

The valve circuit of fig. 2 (*b*) was designed so that its constants would satisfy condition (11). To this end it was arranged that the mutual inductance M_1, between the

TABLE I.

Mutual inductance (henries).	Primary resistance $= R_1$ (ohms).	C_2 μF (actual).	C_2 μF (calculated).
0·186	253	1·020	1·006
	316	1·083	1·068
	376	1·148	1·135
0·1865	253	1·016	1·010
	426	1·209	1·190
	526	1·325	1·294
0·160	253	0·887	0·888
	286	0·916	0·908
	326	0·945	0·935
	376	0·986	0·957
	426	1·027	1·010
	526	1·108	1·102
	626	1·200	1·164
0·104	253	0·720	0·712
	286	0·731	0·720
	326	0·743	0·727
	376	0·761	0·746
	426	0·779	0·762
	526	0·820	0·793
	626	0·862	0·828
0·0913	253	0·690	0·686
	286	0·699	0·692
	326	0·710	0·700
	376	0·724	0·713
	426	0·739	0·726
	526	0·770	0·750
	626	0·803	0·779

secondary coil L_2 and the grid coils L_4 and L_3, could be made zero. As will be seen from the diagram these coils were in series; one of them, L_3, was placed close to L_1, but as far as possible from L_2, while the other was placed close to L_2 but far from L_1. Consequently, even when M_1 was close to zero, the mutual inductance between primary and grid coils was of moderate value.

The E M.F. induced in L_3 and L_4 was amplified by the valve V_2, and a suitable fraction of the amplified voltage was passed to the grid of the valve V_3.

The mutual inductance M was given a value of about 0·145 henry. The capacity C_2 was made zero, the coils L_3 and L_4 were placed in such positions that an alternating current in the coil L_2 induced no E.M.F. in the grid circuit, although the coupling between L_1 and the grid coils L_3 and L_4 was just sufficient to produce oscillations. The mutual inductance between L_1 and L_3 was then somewhat reduced, and the oscillations ceased. C_2 was increased from zero to 0·001 μF, and on making this change the anode current of the valve V_3 increased by about 50 per cent.; further investigation showed that the circuit was oscillating with a frequency which

TABLE II.

Secondary capacity = C_2 (μF).	Observed frequency (cycles per second).	Calculated frequency (cycles per second).
0·001	11,800	14,000
0·011	3,800	4,230
0·100	1,440	1,400

was just above the limits of audition. As C_2 was further increased the frequency of oscillation fell. A small search coil, connected to a cathode ray oscillograph, was placed near L_1, and the approximate frequency of oscillation of the circuit was measured for a few values of the secondary capacity C_2. It may be shown that when C_2 is small and (11) is satisfied the frequency of oscillation of the circuit is approximately given from

$$\omega^2 = 1/(1-k^2)L_2C_2, \quad . \quad . \quad . \quad . \quad . \quad (13)$$

where k is the coefficient of coupling between the primary and secondary coils L_1 and L_2. The results of the measurements made are given in Table II. Under the heading "Calculated frequency" is given the value of the frequency determined from eq. (13).

The poor numerical agreement between theory and observation may be due in part to lack of suitable means

for measuring these frequencies, but I think that it is largely due to stray capacities. The theory, and the fact that a high frequency was generated, seem to provide satisfactory evidence that the frequency was actually higher than the resonant frequency of the secondary.

If the mutual inductance between L_1 and L_3 was slightly greater than the value used in the experiments just described, and C_2 was made zero, the circuit oscillated with a frequency of about 400 cycles per second. On increasing C_2 to any small value greater than $0{\cdot}0005\,\mu F$ the 400 cycle oscillation was replaced by an oscillation of very high frequency, presumably higher than the resonant frequency of the secondary. The non-appearance of this high frequency for values of C_2 less than $0{\cdot}0005\,\mu F$ is considered to be due to deficiencies of the circuit used rather than to any failure of the theory. The circuit used was somewhat unsuitable, particularly in the respect that it was not possible to prevent the flow of grid currents.

Discussion.

On referring to Table I. it will be seen that there is very fair agreement between the observed and calculated values of C_2. In view of discrepancies of greater magnitude than was likely to be caused by uncertainties in the values of the circuit constants it was proposed to make another set of measurements similar to those given above. Preliminary experiments, however, showed that this was unlikely to be profitable. If all the circuit constants but C_2 were fixed, and C_2 was then adjusted to give equal amplitudes of oscillation for both frequencies, it was found that C_2 was not independent of the steady potentials applied to the grids of the valves V_1 and V_2, nor of the current heating the filament of the valve V_1. That is, one finds experimentally, when the circuit is adjusted so that the amplitude of oscillation is the same for either frequency, the relation between the circuit constants is only approximately that given by (12). As the theory takes no account of harmonics which must be present, it cannot be expected that the quantitative results of the theory will be in accurate accord with experiment. The measure of agreement obtained above is to be regarded as showing that theory and experiment are in reasonable accord.

The results recorded in Table I. throw some light on the double-click method of adjusting a circuit so that its resonant frequency is the same as the frequency produced by an oscillator. The tuning is usually regarded as correct if the coupling is weak, *i. e.*, if M is small, and the secondary capacity is given a value midway between the clicks. The values of C_2 recorded in the table must be close to the value of the capacity midway between the clicks, and yet, even for the weakest coupling used, C_2 is still much greater than the value which would give resonance with the oscillator frequency. The true resonance value of C_2, if taken to be the value that makes $L_1C_1=L_2C_2$, is about $0{\cdot}60\,\mu F$. Experimentally the double-click method is known to be unsatisfactory, and the above remarks are of interest in that they indicate, on theoretical grounds, that very serious errors may be made by using the method.

The two circuits studied exhibit properties which are in some respects quite different from those possessed by series arc-maintained circuits. In view of this it is clearly undesirable to use the theory of the series arc circuit to predict the behaviour of coupled valve oscillators *.

Appendix.

If v is written for $v_p+\mu v_g$, the differential equation for the back-coupled oscillator with coupled secondary is found to be

$$\left\{\left(L_1D^2+R_1D+\frac{1}{C_1}\right)\left(L_2D^2+R_2D+\frac{1}{C_2}\right)-M^2D^4\right\}v + \frac{1}{C_1}\left\{(\rho L_1D+R_1)\left(L_2D^2+R_2D+\frac{1}{C_2}\right) - M^2D^3(1-\sigma)\right\}f(v)=0, \quad . \quad . \quad . \quad . \quad (1.1)$$

where D represents the operation of taking a derivative with respect to time,

$$\rho=1-\mu M_0/L_1 \text{ and } \sigma=\mu M_1/M. \quad . \quad . \quad . \quad (1.2)$$

* This is common practice in the theory of some methods of resistance measurement. (*Cf.* Colebrook and Wilmotte, Proc. I. E. E. vol. lxix. p. 497 (1931). In this case the more accurate theory yields a result which is not very different from that given by the simpler arc theory.)

By putting $\rho=1$ and $\sigma=0$ we obtain the equation appropriate to the dynatron oscillator.

If in (1.1) we use alternating current methods, and replace D by $j\omega$ and write a for v and $a\phi(a)$ for $f(v)$, we can readily obtain the results given in (6) and (10) above.

If, however, in (1.1) we put $v=a\sin\omega t$ and $f(v)=a\phi(a)\sin\omega t$, and equate to zero the coefficients of $\sin\omega t$ and $\cos\omega t$, we obtain, if a is not zero,

$$x^2(1-k^2)-x[1+n+\lambda e+R_1\phi(a)\{1+\rho\lambda\}]+n[1+R_1\phi(a)]=0, \quad . \; . \; . \quad (1.3)$$

$$x\{R_1\phi(a)[\rho-k^2(1-\sigma)]+(1+\lambda)e\} = (n+\lambda)e+R_1\phi(a)[n\rho+\lambda e], \quad (1.4)$$

where

$$\omega^2=x/L_1C_1,\ L_1C_1/L_2C_2=n,\ L_1R_2/L_2R_1=\lambda,$$
$$M^2=k^2L_1L_2, \text{ and } R_1{}^2C_1/L_1=e. \; . \; . \; . \quad (1.5)$$

We eliminate x from (1.3) and (1.4), and obtain a cubic in $\phi(a)$, which we write as $Q[\phi(a)]=0 \quad . \; . \; . \quad (1.6)$

$$\begin{aligned} Q[\phi(a)]=&[(n+\lambda)e+R_1\phi(a)\{n\rho+\lambda e\}]^2(1-k^2) \\ &+n[1+R_1\phi(a)][R_1\phi(a)\{\rho-k^2(1-\sigma)\}+(1+\lambda)e]^2 \\ &-[(1+n+\lambda e)+R_1\phi(a)(1+\rho\lambda)][(n+\lambda)e \\ &+R_1\phi(a)\{n\rho+\lambda e\}][R_1\phi(a)\{\rho-k^2(1-\sigma)\}+(1+\lambda)e]. \end{aligned}$$
$$. \; . \; . \quad (1.7)$$

The values of $\phi(a)$ satisfying (1.6) and corresponding to real positive values of amplitude a and frequency ω define possible oscillations of the circuit. Some such possible oscillations are, of course, unstable *.

From eq. (2) and (7) above we see that without a secondary

$$R_1\phi(a)=-e/\rho.$$

It may be shown, following Appleton and van der Pol †, that for stable oscillations in the absence of a secondary $\rho d\phi(a)/da$ is positive. We assume that this is also true

* Limitations of space do not permit this aspect of the problem being discussed here. General stability conditions can be derived by extending van der Pol's (*loc. cit.*) method to any fourth-order differential equation.

† Appleton and van der Pol, Phil. Mag. xlii. p. 201 (1921).

when a secondary is present. Hence, when the secondary increases the amplitude of oscillation of the circuit, it makes $\rho\phi(a)$ greater than $-e/R_1$, the value appropriate to no secondary, or, in other words, it results in (1.6) being satisfied by a value of $R_1\phi(a)$ lying between $-e/\rho$ and zero. From (1.7) we find that $Q[\phi(a)]$ is negative when $\phi(a)=0$. If then $Q[\phi(a)]$ is positive when $R_1\phi(a)=-e/\rho$, (1.6) is satisfied by some value of $R_1\phi(a)$ lying between $-e/\rho$ and zero, and in this case the amplitude of oscillation is increased by the secondary.

The amplitude of oscillation is increased if

$$\frac{e^2k^2}{\rho}\left(1-\frac{e}{\rho}\right)\left\{n\left[\lambda(1-\sigma)+\frac{k^2}{\rho}(1-\sigma)^2\right]-\lambda\left(1-\frac{e}{\rho}\right)(1+\lambda\rho-\sigma)\right\}>0. \quad (1.8)$$

For the dynatron circuit the amplitude of oscillation is increased if *

$$n>\lambda(1-e)(1+\lambda)/(\lambda+k^2) \quad . \quad . \quad . \quad . \quad (1.9)$$

If λ is about unity and k^2 is small, (1.9) is satisfied for values of n rather greater than unity. In special cases, however, λ may be decreased considerably by increasing R_1 and k^2 may be made 0·3 or even more, so that (1.9) may be satisfied for n less than unity. Thus it is possible to have dynatron circuits in which the secondary increases the amplitude of oscillation, even though its resonant frequency is lower than that of the primary.

For the back-coupled oscillator ρ is negative. Thus (1.8) is satisfied if

$$n\left[\lambda(1-\sigma)+\frac{k^2}{\rho}(1-\sigma)^2\right]-\lambda\left(1-\frac{e}{\rho}\right)(1+\lambda\rho-\sigma)<0. \quad (1.10)$$

If $1-\sigma$ is negative (1.10) is satisfied when

$$n>\frac{\lambda\left(1-\frac{e}{\rho}\right)(1+\lambda\rho-\sigma)}{\lambda(1-\sigma)+\frac{k^2}{\rho}(1-\sigma)^2}, \quad . \quad . \quad . \quad (1.11)$$

* For a dynatron circuit capable of producing oscillations $(1-e)$ is positive.

so that once again the amplitude is increased for the higher values of n. If, however,

$$1-\sigma>0 \quad \text{and} \quad \lambda+\frac{k^2}{\rho}(1-\sigma)<0, \quad . \quad . \quad (1.12)$$

(1.11) still gives the condition that the secondary increases the amplitude of oscillation, but as the expression on the right-hand side of (1.11) is now negative (1.11) is satisfied for any value of n. Condition (1.12), together with the requirement that ρ should be negative, is equivalent to condition (11) above.

From (1.3) we see that if n is very large

$$x=n/(1-k^2) \text{ or } x=1+\mathrm{R}_1\phi(a).$$

From (1.4) we find that if $e+\mathrm{R}_1\phi(a)\rho$ differs from zero by a finite amount for n large, x must be of the order of n rather than of the order of unity. Thus when (1.12) and (1.11) are satisfied, and consequently $\mathrm{R}_1\phi(a)<-e/\rho$, for large values of n

$$x=n/(1-k^2), \text{ so that } \omega^2=1/\mathrm{L}_2\mathrm{C}_2(1-k^2). \quad . \quad (1.13)$$

The sign and value of σ are of importance to us. If we write $\mathrm{M}_1^2=k_1^2\mathrm{L}_2\mathrm{L}_3$ and $\mathrm{M}_0^2=k_0^2\mathrm{L}_1\mathrm{L}_3$, and use the fact that $\mathrm{M}^2=k^2\mathrm{L}_1\mathrm{L}_2$, we have

$$\sigma=\mu\mathrm{M}_1/\mathrm{M}=\mu\frac{k_1}{k}\sqrt{\frac{\mathrm{L}_3}{\mathrm{L}_1}}.$$

But

$$\rho=1-\mathrm{M}_0\mu/\mathrm{L}_1=1-\mu k_0\sqrt{\frac{\mathrm{L}_3}{\mathrm{L}_1}}.$$

Hence

$$\sigma=(1-\rho)k_1/kk_0. \quad . \quad . \quad . \quad . \quad (1.14)$$

Now ρ is negative ; hence M_0 and k_0 must be positive. On referring to fig. 1 (*b*) we see that if L_1, L_2, and L_3 are coaxial coils M_1 and M have the same sign *. Hence in this case σ is a positive quantity which is greater than k_1/kk_0. If k, k_0, and k_1 are of the same order of magnitude σ will be considerably greater than unity. More definite information has not been obtained, but it seems reasonable to assume that σ is normally greater than unity. If so, $1-\mu\mathrm{M}_1/\mathrm{M}$ is normally a negative quantity.

* For the method of determining the sign of the mutual inductance *cf.* Nimmo and Poole, Proc. Camb. Phil. Soc. vol. xxviii. p. 531 (1932).

Finally, let us consider the method of determining the condition that the amplitude a, and hence $\phi(a)$, has the same value for both frequencies of oscillation. On turning to (1.4) we see that for a given value of $\phi(a)$, there corresponds one, and only one, value of x, *i. e.*, of the frequency, unless

$$\frac{1+\lambda}{\rho-k^2(1-\sigma)} = \frac{n+\lambda}{n\rho+\lambda e}; \quad . \quad . \quad . \quad (1.15)$$

(1.15) then gives the relation between the circuit constants for which both frequencies have the same amplitude of oscillation.

In concluding, I wish to express my appreciation of the assistance which Mr. H. D Poole has given me during the course of this work. It is largely on account of his help that I have been able to complete the experimental work described in this paper.

SUMMARY.

The paper investigates the effect of a secondary circuit on the amplitude of oscillation of two types of valve oscillator. It is shown that it may be arranged that the presence of the secondary leads to an amplitude larger than that occurring in its absence. The paper also gives details of an investigation, both theoretical and experimental, of the circumstances in which changes in the mutual inductance between a valve oscillator and a coupled secondary are able to bring about discontinuous frequency changes.

XCI. *Certain Vibration Problems solved by means of an Analogous Problem in Heat Conduction.* *By* GEORGE GREEN, *D.Sc., Lecturer in Physics in the Applied Physics Department, The University, Glasgow* *.

THE vibration problems here to be considered belong to a type which occurs in connexion with certain well-known laboratory experiments. Approximate solutions of the problems, sufficient for all that is usually

* Communicated by the Author.

required, appear in many of the textbooks, but exact solutions, so far as the writer is aware, have not been given. The problems present certain difficulties when the solution is attempted by the ordinary methods, so that the solutions here given may therefore be of some interest in themselves. In the method of solution adopted below the treatment of each problem is the same as that employed by the author in previous papers in the solution of heat conduction problems : and the present paper is therefore also intended to illustrate certain remarks made in a former paper * regarding mathematical analogues to be found in diverse subjects in problems involving similar boundary conditions.

Consider first the following heat conduction problem. A large vessel, kept at zero temperature, is connected to a large mass M by a vertical conducting rod of length a, cross-section A, density ρ, specific heat σ, conductivity K. We assume that the mass M is composed of material of specific heat s, and of unit conductivity, so that the temperature is to be regarded as always uniform throughout the mass. The whole system being initially at zero temperature, from a certain instant ($t=0$) onwards a steady supply of heat, Q units per second, is delivered to the mass M. It is required to determine the temperature conditions throughout the conducting rod at any later instant t and the final temperature conditions reached throughout the system, heat losses by radiation and convection being considered negligible.

Let the origin be chosen at the upper end of the connecting rod, at which the temperature is kept at zero. The above problem then involves the conditions—

$$\frac{d\theta}{dt} = \kappa \frac{d^2\theta}{dx^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

the general equation governing heat conduction in the rod together with

$$\theta = 0, \quad \text{initially}, \quad . \quad . \quad . \quad . \quad (2)$$

and, at any later instant,

$$\theta = 0, \quad \text{at } x=0, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\mathrm{M}s\frac{d\theta}{dt} = \mathrm{Q} - \mathrm{KA}\frac{d\theta}{dx}, \quad \text{at } x=a. \quad . \quad (4)$$

* Phil. Mag. xii. Suppl. p. 251 (1931) ; ser. 7, xviii. p. 625 (1934).

On introducing a new temperature function θ_1, defined by

$$\theta = \theta_1 + \frac{Q}{KA}x, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

we find that the conditions determining the function θ_1 are these—the heat conduction equation (1) together with

$$\theta_1 = -\frac{Q}{KA}x, \quad \text{initially,} \quad . \quad . \quad (6)$$

and, at any later instant,

$$\theta_1 = 0, \quad \text{at } x=0, \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$\frac{d\theta_1}{dt} = -\frac{KA}{Ms}\frac{d\theta_1}{dx}, \quad \text{at } x=a. \quad . \quad . \quad (8)$$

The first step in the procedure towards the determination of θ_1 is to find a solution representing the effect of a periodic heat source of strength qe^{ikt} situated at any section $x=x'$ within the rod subject to the end conditions stated. This preliminary problem is indicated by the equations

$$\frac{d^2\theta_1}{dx^2} + \lambda^2\theta_1 = 0, \quad \sqrt{\frac{ik}{\kappa}} = i\lambda; \quad . \quad . \quad . \quad (9)$$

$$\theta_1 = 0, \quad \text{at } x=0; \quad . \quad . \quad . \quad . \quad (10)$$

$$\lambda^2\theta_1 = \frac{KA}{Ms\kappa}\frac{d\theta_1}{dx}, \quad \text{at } x=a. \quad . \quad . \quad (11)$$

The two wave-train solutions of (1),

$$\theta_1 = \rho_1 e^{ikt - i\lambda(x-x')}; \quad \theta_1 = \rho_1 e^{ikt - i\lambda(x'-x)}; \quad \rho_1 = \frac{q}{2Ki\lambda}, \quad (12)$$

taken together satisfy the source condition at $x=x'$ *. Expressions representing the complete effect produced at any point x within the rod by the periodic source at x' are to be found by summing all the wave-train solutions obtained from the positive and negative trains respectively represented in (12) by continued reflexions at the two boundaries $x=0$ and $x=a$. At boundary $x=0$ the negative train is continued by a positive train $\theta_1 = -\rho_1 e^{ikt - i\lambda(x'+x)}$ in order to fulfil condition (7), so that the boundary coefficient at $x=0$ is -1. Similarly at

* Phil. Mag. ix. p. 242 (Feb. 1930).

boundary $x=a$ the positive train in (12) is continued by the negative train $\theta_1 = B\rho_1 e^{ikt - i\lambda(2a-x'-x)}$, where B represents a boundary coefficient so chosen that the condition (8) at $x=a$ is fulfilled. Taking into account all the successive continuing wave-trains we find that the effect produced by the original positive train and its continuations is represented by

$$\theta_1 = \rho_1 e^{ikt - i\lambda(x-x')} f(\lambda, x) \quad . \quad . \quad . \quad . \quad (13)$$

for $x>x'$, and the effect in the same region produced by the original negative train and its continuations is represented by

$$\theta_1 = -\rho_1 e^{ikt - i\lambda(x+x')} f(\lambda, x), \quad . \quad . \quad . \quad . \quad (14)$$

where

$$f(\lambda, x) = \frac{1+Be^{-i\lambda \cdot 2(a-x)}}{1+Be^{-i\lambda \cdot 2a}} \quad . \quad . \quad . \quad . \quad . \quad (15)$$

The total effect produced by the periodic source is represented by the sum of these two expressions,

$$[\theta_1(x, x')] = \rho_1 e^{ikt - i\lambda x} \cdot 2i \sin \lambda x' \cdot f(\lambda, x), \quad (x>x'). \quad (16)$$

In like manner we may show that the total effect produced by the periodic source at any point in the region $(x<x')$ is represented by

$$\theta_1 = [\theta_1(x', x)]. \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Another form of the result (16) is

$$\theta_1 = [\theta_1(x, x')] = 4\rho_1 e^{ikt} \left\{ \frac{\sin \lambda x \sin \lambda x'}{1+Be^{i\lambda \cdot 2a}} + \frac{i}{2} e^{i\lambda x} \sin \lambda x \right\} .$$

$$. \quad . \quad . \quad (16')$$

The next step in the procedure is to determine from (16) and (17) the effect produced at any point by an *instantaneous source of* strength q per unit area situated at the plane $x=x'$, this effect being represented by

$$\theta_1 = \frac{1}{\pi} \int_0^\infty dk\, [\theta_1(x, x')], \quad \text{for } x>x' ; \quad (18)$$

and by the corresponding expression with x and x' interchanged for $x<x'$. The equivalent of (18) in terms of λ is

$$\theta_1 = \frac{2iq\kappa}{\pi K} \int d\lambda\, e^{-\kappa\lambda^2 t} \sin \lambda x' e^{-i\lambda x} f(\lambda, x), \quad x>x', \quad (19)$$

in which the integration is from 0 to ∞ along a line inclined at 45° to the real axis : and we are concerned only with the real part of the integral. The integrand being an odd function of λ, contour integration along a path, consisting of the two lines inclined at $\pm 45°$ respectively to the axis of λ joined by an arc at infinite distance from the origin, yields twice the real value of the required integral. The same integration is, according to the theorem of residues, equivalent to $2\pi i$ times the sum of the residues contained within the contour. By this means the real part of the integral (19) can be evaluated.

The boundary coefficient B is found by taking any two wave-trains such as

$$\theta_1 = \rho_1 e^{ikt}\{e^{-i\lambda(x-x')} + \mathrm{B}e^{-i\lambda(2a-x'-x)}\}, \quad . \quad . \quad (20)$$

and by choosing B so that θ_1 satisfies condition (8) above.

We thus find

$$\mathrm{B} = \frac{i\mathrm{KA} + \mathrm{M}s\kappa\lambda}{i\mathrm{KA} - \mathrm{M}s\kappa\lambda}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

and

$$1 + \mathrm{B}e^{-i\lambda 2a} = 2ie^{-i\lambda a}\{\mathrm{KA}\cos\lambda a - \mathrm{M}s\kappa\lambda\sin\lambda a\}, \quad (22)$$

with a similar expression for the numerator of $f(\lambda, x)$, in which a in (22) is replaced by $(a-x)$. The integral (19) may now be written in the form

$$\theta_1 = \frac{2iq\kappa}{\pi\mathrm{K}} \int d\lambda e^{-\kappa\lambda^2 t} \sin\lambda x' \times \frac{\mathrm{KA}\cos\lambda(a-x) - \mathrm{M}s\kappa\lambda\sin\lambda(a-x)}{\mathrm{KA}\cos\lambda a - \mathrm{M}s\kappa\lambda\sin\lambda a}. \quad (23)$$

The poles of the integrand are at values of λ determined by the positive roots of the equation

$$\tan\lambda a = \frac{\mathrm{KA}}{\mathrm{M}s\kappa\lambda}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

and the evaluation of the real part obtained by means of the residue theorem is given by

$$\theta_1 = \frac{2q\kappa}{\mathrm{K}} \sum_1^\infty e^{-\kappa\lambda^2 t} \sin\lambda x' \sin\lambda x \, \mathrm{A}_n, \quad . \quad . \quad (25)$$

where

$$\mathrm{A}_n = \frac{(\mathrm{KA})^2 + (\mathrm{M}s\kappa\lambda)^2}{a\{(\mathrm{KA})^2 + (\mathrm{M}s\kappa\lambda)^2\} + \mathrm{M}s\kappa\mathrm{KA}}, \quad . \quad . \quad (26)$$

in which the summation applies to all positive roots of equation (24). This result gives the effect of an instantaneous source of strength q at x' for $x>x'$, and also for $x'>x$.

To complete the solution of the problem defined by (1), (6), (7), (8) above, we have yet to take into account the condition (6). Take in (25) $q=\rho\sigma f(x')dx'$, where $f(x)$ represents $-\frac{Q}{K\bar{A}}x$, the initial temperature distribution throughout the rod required by (6), and integrate with respect to x' from $x'=0$ to $x'=a$. This process yields the value of θ_1 required to fulfil conditions (1), (6), (7), (8), in the form

$$\theta_1 = 2\sum_1^\infty e^{-\kappa\lambda^2 t}\sin\lambda x\, A_n \int_0^a dx' \sin\lambda x' f(x'), \quad . \quad (27)$$

in which the coefficient A_n is defined by (26) and the summation applies to all positive roots of equation (24). The solution of the original problem proposed follows at once from (5),

$$\theta=\theta_1+\frac{Q}{K\bar{A}}x \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The solution to our problem at which we have arrived in (27), being a simple summation of sine terms, each term of which satisfies (1), (7), and (8), giving when $t=0$ the required initial value defined by (6), it might be supposed that the ordinary method of obtaining the solution would be simpler than that employed above. If we assume that

$$\theta_1 = \sum_1^\infty e^{-\kappa\lambda^2 t}\sin\lambda x\, B_n, \quad . \quad . \quad . \quad . \quad (28)$$

where λ denotes λ_n and where the summation refers to all positive roots of (24), represents a solution satisfying all requirements, and proceed by the usual method to evaluate the coefficients B_n by means of the condition

$$\theta_1(t=0) = f(x) = \sum_1^\infty \sin\lambda_n x\, B_n, \quad . \quad . \quad . \quad (29)$$

we find that

$$\int_0^a dx' \sin\lambda_m x' \sin\lambda_n x' = -\frac{Ms\kappa}{K\bar{A}}\sin\lambda_m a \sin\lambda_n a, \quad (30)$$

where λ_m and λ_n are two different roots of equation (24) and that

$$\int_0^a dx' \sin^2 \lambda_m x' = \frac{1}{2}\left\{ a - \frac{\sin \lambda_m a \cos \lambda_m a}{\lambda_m} \right\}. \quad (31)$$

Hence

$$\int_0^a dx' \sin \lambda_m x' f(x')$$

$$= -\frac{Ms\kappa}{KA} \sin \lambda_m a \sum_{1 \; n \neq m}^{\infty} \sin \lambda_n a \, B_n + B_m \int_0^a dx' \sin^2 \lambda_m x' \quad (32)$$

and

$$B_m = \frac{\int_0^a dx' \sin \lambda_m x' f(x') + \frac{Ms\kappa}{KA} \sin \lambda_m a \sum_1^{\infty} \sin \lambda_n a \, B_n}{\frac{Ms\kappa}{KA} \sin^2 \lambda_m a + \int_0^a dx' \sin^2 \lambda_m x'} \quad .* (33)$$

The result here obtained should be in complete agreement with (27) above ; but it appears that (33) serves to confirm (27) only provided that

$$\sum_1^{\infty} \sin \lambda_n a \, B_n = 0. \quad . \quad . \quad . \quad . \quad . \quad (34)$$

The proof of this would necessitate a special mathematical investigation, and there is no doubt that the result given in (27) is correct.

Consider now the following problem involving elastic vibrations. A mass M hangs in equilibrium at the end of a vertical wire of length a, cross-section A, density ρ, Young's modulus E, whose upper end is fixed to a rigid support. At a certain instant ($t=0$) the mass is raised so as just to relieve the wire from strain, and the system is then left free to vibrate. Determine the subsequent motion.

Let the origin be taken at the fixed upper end of the wire ; let x be the distance from the origin of any horizontal layer of particles in the wire in the unstrained condition, and let u be the downward displacement of the same layer of particles at instant t. Assuming that the weight of the wire itself may be neglected, we may

* A case similar to this has been treated by J. Robertson, Phil. Mag. xv. p. 948 (May 1933).

then indicate the above problem by the mathematical conditions—

$$\frac{d^2u}{dt^2} = V^2 \frac{d^2u}{dx^2}, \qquad V^2 = \frac{E}{\rho}, \quad . \quad . \quad . \quad . \quad (35)$$

$$u = 0, \quad \text{initially} \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

and at any later instant,

$$u = 0, \quad \text{at } x = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

$$Mu = Mg - EA \frac{du}{dx}, \quad \text{at } x = a. \quad . \quad . \quad (38)$$

These equations correspond with (1) to (4) above. When we introduce a new variable, u_1, as in (5), defined by

$$u = u_1 + \frac{Mg}{EA} x, \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

we find that the conditions determining u_1 are represented by (35) together with

$$u_1 = -\frac{Mg}{EA} x, \quad \text{initially}, \quad . \quad . \quad (40)$$

and at any later instant,

$$u_1 = 0, \quad \text{at } x = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

$$\ddot{u}_1 = -\frac{EA}{M} \frac{du_1}{dx}, \quad \text{at } x = a. \quad . \quad . \quad (42)$$

These conditions correspond with (1), (6), (7), (8) above.

From this point onwards the procedure adopted in determining u_1 corresponds exactly with that followed in determining θ_1 in the former problem. The expression representing the effect of a periodic displacement, $u_1 = e^{ikt}$, located at any section $x = x'$, is constructed, as before, from the fundamental wave-train solutions of (35),

$$e^{ikt - i\frac{k}{V}(x - x')} \quad \text{and} \quad e^{ikt - i\frac{k}{V}(x' - x)}, \quad . \quad . \quad (43)$$

V being the wave-velocity common to all trains in this case. For convenience of comparison of the present problem with that previously considered we here put down side by side the corresponding conditions for each:

$$\frac{d^2u_1}{dx^2} + \frac{k^2}{V^2} \ddot{u}_1 = 0, \qquad \frac{d^2\theta_1}{dx^2} + \lambda^2 \theta_1 = 0, \quad \text{at } (x, t); \quad (44)$$

$$u_1 = -\frac{Mg}{EA} x, \qquad \theta_1 = -\frac{Q}{KA} x, \qquad \text{at } t = 0; \quad (45)$$

$$u_1 = 0, \qquad \theta_1 = 0, \qquad \text{at } x=0\,; \quad (46)$$

$$\frac{k^2}{V^2}u_1 = \frac{EA}{MV^2}\cdot\frac{du_1}{dx}, \quad \lambda^2\theta_1 = \frac{KA}{Ms\kappa}\frac{d\theta_1}{dx}, \quad \text{at } x=a. \quad (47)$$

The fundamental wave-trains, (43), correspond exactly with (12) above, ρ_1 being replaced by unity and λ^2 by (k^2/V^2). With the aid of the same substitutions we can obtain from (16) and (17) above expressions representing the effect of a periodic displacement $u_1=e^{ikt}$ located at $x=x'$. The corresponding expression representing the effect of an instantaneous or concentrated unit displacement located at $x=x'$, in accordance with the well-known Fourier integral theorem upon which the whole process employed is based, may be written in the form

$$u_1 = \frac{2}{V}\frac{i}{\pi}\int_0^\infty dk\, e^{ikt}\sin\frac{k}{V}x'\, e^{-i\frac{k}{V}x} f\left(\frac{k}{V}, x\right); \quad . \quad (48)$$

or it may be derived from (18) or (19). The system of wave-trains composing the integral (48) is the same as that composing the integral (19), and the equations {(20), (21), (22)} referring to the boundary coefficient B correspond exactly in the two problems—the constants (EA) and (MV^2) in the elastic problem taking the place of (KA) and ($Ms\kappa$) respectively in the heat problem, as in (47) above. Another form of the same result, which corresponds with (23) above, is

$$u_1 = \frac{2}{V}\frac{i}{\pi}\int_0^\infty dk\, e^{ikt}\sin\lambda x' \times \frac{EA\cos\lambda(a-x) - MV^2\lambda\sin\lambda(a-x)}{EA\cos\lambda a - MV^2\lambda\sin\lambda a}, \quad (49)$$

where $\lambda=(k/V)$. Evaluation of the real part of this integral yields, finally,

$$u_1 = 2\sum_1^\infty \cos kt\sin\frac{k}{V}x'\sin\frac{k}{V}x\,A_n, \quad . \quad . \quad (50)$$

where

$$A_n = \frac{(EA)^2+(MVk)^2}{a\{(EA)^2+(MVk)^2\}+MV^2EA}, \quad . \quad . \quad (51)$$

and the summation applies to all positive roots of

$$\tan\frac{k}{V}a = \frac{EA}{MVk}. \quad . \quad . \quad . \quad . \quad . \quad (52)$$

To complete the solution of the original problem proposed multiply (50) by $f(x')dx'$, where $f(x)$ represents $-\frac{Mg}{EA}x$, the initial displacement throughout the rod, and integrate from $x'=0$ to $x'=a$. This yields, in virtue of (39),

$$u=\left[\frac{Mg}{EA}x+2\sum_{1}^{\infty}\cos kt\,\sin\frac{k}{V}x\,A_n\int_0^a dx'\,\sin\frac{k}{V}x'f(x')\right], \quad (53)$$

where A_n is given by (51), and the summation applies to all positive roots of (52).

The analysis employed above and the results obtained may be applied in like manner to the solution of the corresponding problem involving torsional vibrations, which may be stated as follows.

A uniform wire of length a, radius R, rigidity modulus n, fixed at its upper end, hangs vertically, and carries at its lower end a symmetrical mass M whose moment of inertia about an axis coinciding with the axis of the wire is I. Assuming that the mass M is turned about the axis of the wire until the angle of twist at the lower end is $-\theta_0$ and the system is then left free to vibrate, determine the subsequent motion.

Taking θ to represent the angle of twist at section x measured from the fixed end, at instant t from the start of motion, we may indicate the above problem by the conditions

$$I\frac{d^2\theta}{dt^2}=\frac{n}{\rho}\frac{d^2\theta}{dx^2}; \quad \text{at } (x,\,t), \quad . \quad . \quad (54)$$

$$\theta=-\frac{\theta_0}{a}x, \quad \text{initially}, \quad . \quad . \quad (55)$$

and, at any later instant,

$$\theta=0, \quad \text{at } x=0, \quad . \quad . \quad . \quad . \quad (56)$$

$$I\ddot{\theta}=-\tfrac{1}{2}\pi n R^4\frac{d\theta}{dx}, \quad \text{at } x=a. \quad . \quad (57)$$

This problem is obviously an exact analogue of that of determining u_1 to satisfy conditions (35), (40), (41), (42) above. The solution may accordingly be written down from the result given in (53).

XCII. *The Stress in a Twisted Circular Cylinder having Circular Holes.* *By* Masaiti Kondô, *Professor of the Yokohama Technical College, Yokohama, Japan.**

I. *General Description.*

SINCE the ordinary methods of conformal transformations etc. used for solving various torsion problems were not suitable for such complicated sections as in the present paper, the author adopted the scheme as described below.

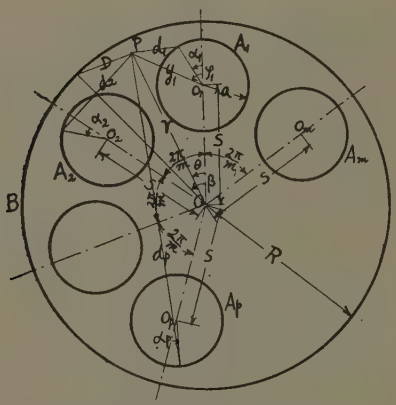
Analogies with hydrodynamics show us that the real part w in $f(z)=f(x+iy)=w+i\psi$ corresponds to the velocity potential of frictionless liquid and to the axial displacement of torsion problem; the imaginary part ψ corresponds to the stream function of the former as well as to the stress function of the latter. Now, when an elementary circulation Γds exists at a point z_0 (ds meaning the elementary length through z_0), $f(z)$ is given by $(1/2\pi)i\Gamma\, ds\, \ln(z-z_0)$, and the stream function by $d\psi=(1/2\pi)\Gamma\, ds\, \ln r$, where r denotes the distance of a point z from z_0. The author assumes, therefore, an elementary stress function in the form $d\psi=(G\tau/2)RC\, ds\, \ln r$, where G denotes the rigidity, τ the angle of torsion per unit length, R a linear length (in this case the outer radius of the cylinder). C, a dimensionless quantity, is defined here as the "strength of stress function" at that point. The stress function in this form of course satisfies the fundamental equation of torsion problem, and our next step is to make it satisfy the boundary conditions. For this purpose the author distributes the strength of stress function C over the section in some functional forms, and determines them so that the stress function $\psi=\int d\psi=(G\tau/2)R\int C\, ds\, \ln r$ may satisfy $\psi-\frac{G\tau}{2}r^2=\text{const.}$ along the corresponding boundaries. This procedure may be called the "stress function distribution method."

This conception is not new, and has been used by many authors with good results for the allied problems of other theoretical fields, *e. g.*, potential flow around an airship

* Communicated by Prof. S. Yokota.

body * etc., theory of wing of finite span †, wall interference of wind tunnel ‡, theories of heat conduction and of electric field §. The shear stress in a twisted circular cylinder having a circular hole (concentric or eccentric) is well known. The present cases having two or more holes are important from the technical point of view. It is interesting also to see that similar problems in hydrodynamics, heat conduction, and electric potential can be foreseen from the present results, and that the procedure presented here seems to promise further applications.

Fig. 1.



Circular cylinder having m circular holes.

II. *Mathematical Treatment.*

Consider a circular cylinder of radius R, with its centre at O, having m circular holes of radius a at symmetrical positions with respect to O, the centres of which being at O_1, O_2, ..., O_p, ..., O_m, with eccentricity s. Call the

* Th. v. Kármán, *Abhand. a. d. Aerodyn. Inst. a. d. Tech. Hochsch. Aachen*, H. 6 (1927); K. Kubo, Jour. of Aeron. Resear. Inst., Tôkyô Imp. Univ., Nov. 1931.

† H. Glauert, 'The Elements of Aerofoil and Airscrew Theory,' chap. xi. (1930).

‡ M. Sanuki and I. Tani, Proc. Phys.-Math. Soc. Japan, Nov. 1932.

§ K. Miyagi, Jour. of Inst. of Elect. Engr. of Japan, Oct. 1932 and Apr. 1933; J. F. Böttcher, *Z. A. M. M.*, Oct. 1934.

inner circles A_1, A_2, ..., A_p, ..., A_m and the outer one B. Other notations are shown in fig. 1.

Now we distribute the strength of stress function C along the inner and outer circles A_1, A_2, ..., and B in the Fourier's serial forms as follows :—

$$C_{A_1} = \sum_{n=1}^{\infty} A_n \cos n\alpha_1, \quad C_{A_2} = \sum_{n=1}^{\infty} A_n \cos n\alpha_2, \quad \ldots\ldots\ldots\ldots,$$

$$C_{A_p} = \sum_{n=1}^{\infty} A_n \cos n\alpha_p, \quad \ldots\ldots\ldots\ldots, \quad C_{A_m} = \sum_{n=1}^{\infty} A_n \cos n\alpha_m;$$

$$C_B = \sum_{n=1}^{\infty} B_{mn} \cos mn\beta,$$

where A_n, B_{mn} are assumed coefficients, which shall be determined later by the boundary conditions. A_0, B_0 are found to be unnecessary.

1. *ψ at Any Point (r, θ).*

The stress function ψ at any point (r, θ) in the section is the sum of integrations of the elementary stress function; therefore

$$\psi_{PA_1A_2 \ldots A_p \ldots A_m B} = \sum_{p=1}^{m} \psi_{PA_p} + \psi_{PB},$$

where the first suffix means the point at which the stress function is required and each of the succeeding ones the boundary along which the strength of stress function is distributed.

$$\psi_{PA_1} = \int d\psi_{PA_1} = \frac{G\tau}{2} Ra \int_0^{2\pi} \sum_{n=1}^{\infty} A_n \cos n\alpha_1 . \ln d_1 . d\alpha_1,$$

but

$$\ln d_1 = \ln \frac{d_1}{y_1} + \ln y_1 = -\sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{a}{y_1}\right)^k . \cos k(\varphi_1 - \alpha_1)^{[C]} + \ln y_1, \quad a < y_1$$

$$r > s = -\sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{a}{r}\right)^k \left[\cos k\alpha_1 \left\{ \cos k\theta + \sum_{l=1}^{\infty} \frac{k+l-1}{k-1} \left(\frac{s}{r}\right)^l . \cos (k+l)\theta \right\} \right]^{[M]}$$

[C] The expansion marked [C] throughout this paper is by H. L. Curtis's formula (Scient. Pap. of Bur. of Stand., no. 374, Apr. 1920).

[M] The expansions marked [M] throughout this paper are by K. Miyagi's formulæ (*loc. cit.*).

$$+\sin k\alpha_1 \Big\{ \sin k\theta + \sum_{l=1}^{\infty} \frac{|k+l-1}{|k-1\ \ l} \Big(\frac{s}{r}\Big)^l . \sin (k+l)\theta \Big\}^{[M]} \Big] + \ln y_1,$$

$$r<s = -\sum_{k=1}^{\infty} \frac{1}{k} \Big(\frac{a}{s}\Big)^k (-1)^k \times \Big[\cos k\alpha_1 \Big\{ 1 + \sum_{l=1}^{\infty} \frac{|k+l-1}{|k-1\ \ l} \Big(\frac{r}{s}\Big)^l . \cos l\theta \Big\}^{[M]} - \sin k\alpha_1 \sum_{l=1}^{\infty} \frac{|k+l-1}{|k-1\ \ l} \Big(\frac{r}{s}\Big)^l . \sin l\theta^{[M]} \Big] + \ln y_1.$$

Therefore for $r>s$, after integration,

$$\psi_{PA_1} = -\frac{G\tau}{2} Ra\pi \sum_{k=1}^{\infty} \frac{A_k}{k} \Big(\frac{a}{r}\Big)^k \times \Big[\cos k\theta + \sum_{l=1}^{\infty} \frac{|k+l-1}{|k-1\ \ l} \Big(\frac{s}{r}\Big)^l . \cos (k+l)\theta \Big],$$

and similarly for $r<s$,

$$\psi_{PA_1} = -\frac{G\tau}{2} Ra\pi \sum_{k=1}^{\infty} \frac{A_k}{k} \Big(\frac{a}{s}\Big)^k (-1)^k \times \Big[1 + \sum_{l=1}^{\infty} \frac{|k+l-1}{|k-1\ \ l} \Big(\frac{r}{s}\Big)^l . \cos l\theta \Big].$$

Putting $\theta = (p-1)\frac{2\pi}{m} - \theta$ in ψ_{PA_1}, we get

$r>s$

$$\psi_{PA_p} = -\frac{G\tau}{2} Ra\pi \sum_{k=1}^{\infty} \frac{A_k}{k} \Big(\frac{a}{r}\Big)^k \Big[\cos k\Big(\overline{p-1}\,\frac{2\pi}{m} - \theta\Big) + \sum_{l=1}^{\infty} \frac{|k+l-1}{|k-1\ \ l} \Big(\frac{s}{r}\Big)^l . \cos (k+l)\Big(\overline{p-1}\,\frac{2\pi}{m} - \theta\Big)\Big],$$

$r<s$,

$$\psi_{PA_p} = -\frac{G\tau}{2} Ra\pi \sum_{k=1}^{\infty} \frac{A_k}{k} \Big(\frac{a}{s}\Big)^k (-1)^k \times \Big[1 + \sum_{l=1}^{\infty} \frac{|k+l-1}{|k-1\ \ l} \Big(\frac{r}{s}\Big)^l . \cos l\Big(\overline{p-1}\,\frac{2\pi}{m} - \theta\Big)\Big].$$

Therefore, for $r>s$

$$\sum_{p=1}^{m}\psi_{\mathrm{PA}_p}=-\frac{\mathrm{G}\tau}{2}\mathrm{R}a\pi\sum_{k=1}^{\infty}\frac{\mathrm{A}_k}{k}\left(\frac{a}{r}\right)^k\left[\sum_{p=1}^{m}\cos k\left(\overline{p-1}\,\frac{2\pi}{m}-\theta\right)\right.$$
$$\left.+\sum_{l=1}^{\infty}\frac{\lfloor k+l-1}{\lfloor k-1\,\lfloor l}\left(\frac{s}{r}\right)^l\sum_{p=1}^{m}\cos(k+l)\left(\overline{p-1}\,\frac{2\pi}{m}-\theta\right)\right].$$

By the relations

$$\sum_{p=1}^{m}\cos k\left(\overline{p-1}\,\frac{2\pi}{m}-\theta\right)=m\cos k\theta$$

when $k=$ multiples of m, and

$$\sum_{p=1}^{m}\cos k\left(\overline{p-1}\,\frac{2\pi}{m}\right)-\theta=0$$

when $k\neq$ multiples of m, we get

$$\sum_{p=1}^{m}\psi_{\mathrm{PA}_p}=-\frac{\mathrm{G}\tau}{2}\mathrm{R}a\pi\left[\sum_{k=\text{multiples of }m}^{\infty}\frac{\mathrm{A}_k}{k}\left(\frac{a}{r}\right)^k m.\cos k\theta\right.$$
$$\left.+\sum_{k=1}^{\infty}\frac{\mathrm{A}_k}{k}\left(\frac{a}{r}\right)^k\sum_{l=1}^{\infty}\frac{\lfloor k+l-1}{\lfloor k-1\,\lfloor l}\left(\frac{s}{r}\right)^l m.\cos(k+l)\theta\right],$$

where $(k+l)$ must be taken only in the multiples of m. Rewriting under consideration of this limitation, we have for $r>s$

$$\sum_{p=1}^{m}\psi_{\mathrm{PA}_p}=-\frac{\mathrm{G}\tau}{2}\mathrm{R}.ma\pi\left[\sum_{k=1}^{\infty}\frac{\mathrm{A}_{mk}}{mk}\left(\frac{a}{r}\right)^{mk}.\cos mk\theta\right.$$
$$\left.+\sum_{k=1}^{\infty}\sum_{l=1}^{mk-1}\mathrm{A}_l\left(\frac{a}{r}\right)^l\frac{\lfloor mk-1}{\lfloor l\,\lfloor mk-l}\left(\frac{s}{r}\right)^{mk-l}.\cos mk\theta\right].$$

Similarly for $r<s$

$$\sum_{p=1}^{m}\psi_{\mathrm{PA}_p}=-\frac{\mathrm{G}\tau}{2}\mathrm{R}.ma\pi\sum_{k=1}^{\infty}\frac{\mathrm{A}_k}{k}\left(\frac{a}{s}\right)^k(-1)^k$$
$$\times\left[1+\sum_{l=1}^{\infty}\frac{\lfloor k+ml-1}{\lfloor k-1\,\lfloor ml}\left(\frac{r}{s}\right)^{ml}.\cos ml\theta\right].$$

Next as for ψ_{PB},

$$\psi_{\mathrm{PB}}=\int d\psi_{\mathrm{PB}}=\frac{\mathrm{G}\tau}{2}\mathrm{R}^2\int_0^{2\pi}\sum_{n=1}^{\infty}\mathrm{B}_{mn}\cos mn\beta.\ln\mathrm{D}.d\beta,$$

where

$$\ln D = \ln \frac{D}{R} + \ln R = -\sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{r}{R}\right)^k \cos k(\beta-\theta)^{[C]} + \ln R$$

$r < R$.

Therefore by integrating

$$\psi_{PB} = -\frac{G\tau}{2} R^2 \pi \sum_{k=1}^{\infty} \frac{B_{mk}}{mk}\left(\frac{r}{R}\right)^{mk} . \cos mk\theta.$$

Summing up, we get at last

$r > s$

$$\psi_{PA_1A_2\ldots A_p\ldots A_mB} = -\frac{G\tau}{2} R . ma\pi \left[\sum_{k=1}^{\infty} \frac{A_{mk}}{mk}\left(\frac{a}{r}\right)^{mk} . \cos mk\theta + \sum_{k=1}^{\infty} \sum_{l=1}^{mk-1} A_l \left(\frac{a}{r}\right) \frac{\lfloor mk-1}{\lfloor l \, \lfloor mk-l}\left(\frac{s}{r}\right)^{mk-l} . \cos mk\theta \right] - \frac{G\tau}{2} R^2 \pi \sum_{k=1}^{\infty} \frac{B_{mk}}{mk}\left(\frac{r}{R}\right)^{mk} . \cos mk\theta, \quad . \quad . \quad . \quad (I)$$

$r < s$

$$\psi_{PA_1A_2\ldots A_p\ldots A_mB} = -\frac{G\tau}{2} R . ma\pi \sum_{k=1}^{\infty} \frac{A_k}{k}\left(\frac{a}{s}\right)^k (-1)^k \times \left[1 + \sum_{l=1}^{\infty} \frac{\lfloor k+ml-1}{\lfloor k-1 \, \lfloor ml}\left(\frac{r}{s}\right)^{ml} . \cos ml\theta \right] - \frac{G\tau}{2} R^2 \pi \sum_{k=1}^{\infty} \frac{B_{mk}}{mk}\left(\frac{r}{R}\right)^{mk} . \cos mk\theta. \quad (I')$$

2. *ψ on the Outer Circle* (R, θ).

ψ at any point (R, θ) on the outer circle is given by

$$\psi_{BA_1A_2\ldots A_p\ldots A_mB} = \sum_{p=1}^{m} \psi_{BA_p} + \psi_{BB}.$$

Here $\sum_{p=1}^{m} \psi_{BA_p}$ is at once obtained by putting $r = R$ in $\sum_{p=1}^{m} \psi_{PA_p}$, but ψ_{BB} needs some consideration because $\ln D$ in the course of calculation of ψ_{PB} becomes divergent for $r = R$. In this case

$$\ln D = \ln\left(2R \sin \frac{|\beta-\theta|}{2}\right),$$

and accordingly

$$\psi_{BB}=\frac{G\tau}{2}R^3\int_0^{2\pi}\sum_{n=1}^{\infty}B_{mn}\cos mn\beta\,.\ln\sin\frac{|\beta-\theta|}{2}\,.d(\beta-\theta)$$

$$=-\frac{G\tau}{2}R^2\pi\sum_{k=1}^{\infty}\frac{B_{mk}}{mk}\,.\cos mk\theta.$$

Therefore

$$\psi_{BA_1A_2\ldots A_p\ldots A_mB}=-\frac{G\tau}{2}R\,.ma\pi\left[\sum_{k=1}^{\infty}\frac{A_{mk}}{mk}\left(\frac{a}{R}\right)^{mk}\,.\cos mk\theta\right.$$

$$\left.+\sum_{k=1}^{\infty}\sum_{l=1}^{mk-1}A_l\left(\frac{a}{R}\right)^{l}\frac{\lfloor mk-1}{\lfloor l\ \lfloor mk-l}\left(\frac{s}{R}\right)^{mk-l}\,.\cos mk\theta\right]$$

$$-\frac{G\tau}{2}R^2\pi\sum_{k=1}^{\infty}\frac{B_{mk}}{mk}\,.\cos mk\theta. \qquad \ldots\ldots \quad \text{(II)}$$

3. ψ *on the Inner Circle* (a, φ).

ψ at any point (a, φ) on the inner circle A_1, A_2, ..., A_p, ..., A_m (fig. 2), is given by

$$\psi_{A_1A_1A_2\ldots A_p\ldots A_mB}=\psi_{A_1A_1}+\sum_{p=2}^{m}\psi_{A_1A_p}+\psi_{A_1B}.$$

Here $\psi_{A_1A_1}$ is obtained similarly for ψ_{BB} as

$$\psi_{A_1A_1}=-\frac{G\tau}{2}Ra\pi\sum_{k=1}^{\infty}\frac{A_k}{k}\cos k\varphi_1,$$

but the remaining two need some consideration. Referring to fig. 2,

$$\psi_{A_1A_p}=\frac{G\tau}{2}Ra\int_0^{2\pi}C_{A_p}\ln d_p\,.d\alpha_p,$$

but

$$\ln d_p=\ln\frac{d_p}{y_p}+\ln y_p=-\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{a}{y_p}\right)^k\,.\cos k(\pi-\alpha_p+\varphi_p)^{[C]}+\ln y_p$$

$$=-\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\left(\frac{a}{y_p}\right)^k\left[\cos k\left(\alpha_p-\frac{\pi}{2}\left(1-\frac{2\overline{p-1}}{m}\right)\right).\cos k\delta_p\right.$$

$$\left.+\sin k\left(\alpha_p-\frac{\pi}{2}\left(1-\frac{2\overline{p-1}}{m}\right)\right).\sin k\delta_p\right]+\ln y_n$$

$$=-\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\left[\frac{a}{2s\sin\dfrac{\overline{p-1}}{m}\pi}\right]^k\left\{\cos k\left(\alpha_p-\frac{\pi}{2}\right.\right.$$

$$\times\left(1-\frac{2p-1}{m}\right)\Big)\Big[1+\sum_{l=1}^{\infty}(-1)^l\frac{\lfloor k+l-1}{\lfloor k-1\ \lfloor l}$$

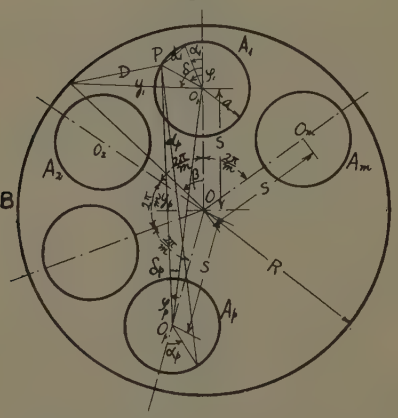
$$\times\left[\frac{a}{2s\sin\dfrac{p-1}{m}\pi}\right]^l.\cos l\left(\varphi_1+\frac{\pi}{2}\left(1-\frac{2p-1}{m}\right)\right)\Big]^{[M]}$$

$$-\sin k\left(\alpha_p-\frac{\pi}{2}\left(1-\frac{2p-1}{m}\right)\right)\sum_{l=1}^{\infty}(-1)^l\frac{\lfloor k+l-1}{\lfloor k-1\ \lfloor l}$$

$$\times\left[\frac{a}{2s\sin\dfrac{p-1}{m}\pi}\right]^l.\sin l\left(\varphi_1+\frac{\pi}{2}\right.$$

$$\left.\times\left(1-\frac{2p-1}{m}\right)\right)^{[M]}\Big\}+lny_p,$$

Fig. 2.



Circular cylinder having m circular holes.

therefore after integration

$$\psi_{A_1A_p}=-\frac{G\tau}{2}Ra\pi\sum_{k=1}^{\infty}(-1)^k\frac{A_k}{k}\left[\frac{a}{2s\sin\dfrac{p-1}{m}\pi}\right]^k$$

$$\times\left\{\left[1+\sum_{l=1}^{\infty}(-1)^l\frac{\lfloor k+l-1}{\lfloor k-1\ \lfloor l}\left[\frac{a}{2s\sin\dfrac{p-1}{m}\pi}\right]^l\right.\right.$$

$$\cdot\cos l\left(\varphi_1+\frac{\pi}{2}\left(1-\frac{2p-1}{m}\right)\right)\Big]$$

$$\cdot\cos k\frac{\pi}{2}\left(1-\frac{2p-1}{m}\right)+\sum_{l=1}^{\infty}(-1)^l\frac{\lfloor k+l-1}{\lfloor k-1\rfloor l}\left(\frac{a}{2s\sin\dfrac{p-1}{m}\pi}\right)^l$$

$$\cdot\sin l\left(\varphi_1+\frac{\pi}{2}\left(1-\frac{2p-1}{m}\right)\right)\cdot\sin k\frac{\pi}{2}\left(1-\frac{2p-1}{m}\right)\Big\}.$$

Accordingly

$$\sum_{p=2}^{m}\psi_{A_1Ap}=-\frac{G\tau}{2}\,Ra\pi\sum_{k=1}^{\infty}(-1)^k\frac{A_k}{k}\sum_{p=2}^{m}\left(\frac{a}{2s\sin\dfrac{p-1}{m}\pi}\right)^k$$

$$\times\Bigg\{\cos k\frac{\pi}{2}\left(1-\frac{2p-1}{m}\right)$$

$$+\sum_{l=1}^{\infty}(-1)^l\frac{\lfloor k+l-1}{\lfloor k-1\rfloor l}\left(\frac{a}{2s\sin\dfrac{p-1}{m}\pi}\right)^l$$

$$\times\Big[\cos(k-l)\frac{\pi}{2}\left(1-\frac{2p-1}{m}\right)\cdot\cos l\varphi_1$$

$$+\sin(k-l)\frac{\pi}{2}\left(1-\frac{2p-1}{m}\right)\cdot\sin l\varphi_1\Big]\Bigg\}.$$

In this expression the relation

$$\sum_{p=2}^{m}\left(\frac{1}{\sin\dfrac{p-1}{m}\pi}\right)^{k+l}\cdot\sin(k-l)\frac{\pi}{2}\left(1-\frac{2p-1}{m}\right)=0$$

exists, so that

$$\sum_{p=2}^{m}\psi_{A_1Ap}=-\frac{G\tau}{2}\,Ra\pi\sum_{k=1}^{\infty}(-1)^k\frac{A_k}{k}$$

$$\times\sum_{p=2}^{m}\left(\frac{a}{2s\sin\dfrac{p-1}{m}\pi}\right)^k\Big[\cos k\frac{\pi}{2}\left(1-\frac{2p-1}{m}\right)$$

$$+\sum_{l=1}^{\infty}(-1)^l\frac{\lfloor k+l-1}{\lfloor k-1\,\lfloor l}\left(\frac{a}{2s\sin\dfrac{p-1}{m}\pi}\right)^l$$

$$.\cos(k-l)\frac{\pi}{2}\left(1-\frac{2p-1}{m}\right).\cos l\varphi_1\Big].$$

Similarly

$$\psi_{A_1B}=\frac{G\tau}{2}R^2\int_0^{2\pi}C_B\ln D.d\beta,$$

where

$$\ln D=\ln\frac{D}{y_1}+\ln\frac{y_1}{R}+\ln R.$$

But

$$\ln\frac{D}{y_1}=-\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{a}{y_1}\right)^k\cos k(\delta-\varphi_1)^{[C]}, \qquad a<y,$$

$$=-\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{a}{R}\right)^k\Big\{\cos k\varphi_1\Big[\cos k\beta$$

$$+\sum_{l=1}^{\infty}\frac{\lfloor k+l-1}{\lfloor k-1\,\lfloor l}\left(\frac{s}{R}\right)^l.\cos(k+l)\beta\Big]^{[M]}$$

$$+\sin k\varphi_1\Big[\sin k\beta+\sum_{l=1}^{\infty}\frac{\lfloor k+l-1}{\lfloor k-1\,\lfloor l}\left(\frac{s}{R}\right)^l.\sin(k+l)\beta\Big]^{[M]}\Big\}, \qquad R>s,$$

and

$$\ln\frac{y_1}{R}=-\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{s}{R}\right)^k\cos k\beta^{[C]}, \qquad R>s.$$

Therefore, after integration,

$$\psi_{A_1B}=-\frac{G\tau}{2}R^2\pi\Big[\sum_{k=1}^{\infty}\frac{B_{mk}}{mk}\left(\frac{a}{R}\right)^{mk}.\cos mk\varphi_1$$

$$+\sum_{k=1}^{\infty}B_{mk}\sum_{l=1}^{mk-1}\left(\frac{a}{R}\right)^l\frac{\lfloor mk-1}{\lfloor l\,\lfloor mk-l}\left(\frac{s}{R}\right)^{mk-l}$$

$$.\cos l\varphi_1+\sum_{k=1}^{\infty}\frac{B_{mk}}{mk}\left(\frac{s}{R}\right)^{mk}\Big].$$

Summing up,

$$\psi_{A_1A_1A_2\ldots Ap\ldots AmB} = -\frac{G\tau}{2} Ra\pi \left\{ \sum_{k=1}^{\infty} \frac{A_k}{k} \cos k\varphi_1 + \sum_{k=1}^{\infty} (-1)^k \frac{A_k}{k} \right.$$

$$\times \sum_{p=2}^{m} \left(\frac{a}{2s \sin \frac{p-1}{m}\pi} \right)^k \left[\cos k\frac{\pi}{2}\left(1 - \frac{2p-1}{m}\right) \right.$$

$$+ \sum_{l=1}^{\infty} (-1)^l \frac{k+l-1}{\underline{k-1}\,\underline{l}} \left(\frac{a}{2s \sin \frac{p-1}{m}\pi} \right)^l$$

$$\left. \left. \cdot \cos (k-l)\frac{\pi}{2}\left(1 - \frac{2p-1}{m}\right) \cdot \cos l\varphi_1 \right] \right\}$$

$$- \frac{G\tau}{2} R^2\pi \left[\sum_{k=1}^{\infty} \frac{B_{mk}}{mk} \left(\frac{a}{R}\right)^{mk} \cdot \cos mk\varphi_1 \right.$$

$$+ \sum_{k=1}^{\infty} B_{mk} \sum_{l=1}^{mk-1} \left(\frac{a}{R}\right)^l \frac{\underline{mk-1}}{\underline{l}\;\underline{mk-l}} \left(\frac{s}{R}\right)^{mk-l} \cdot \cos l\varphi_1$$

$$\left. + \sum_{k=1}^{\infty} \frac{B_{mk}}{mk} \left(\frac{s}{R}\right)^{mk} \right] \ldots\ldots\ldots\ldots \text{(III)}$$

4. *Boundary Conditions and Determination of Assumed Coefficients.*

The necessary boundary condition along the outer circle is

$$\psi_{BA_1A_2\ldots Ap\ldots A_mB} - \frac{G\tau}{2} R^2 = \text{independent of } \theta, \textit{ i. e.},$$

$$- \frac{G\tau}{2} R\pi \sum_{k=1}^{\infty} \left\{ ma \left[\frac{A_{mk}}{mk} \left(\frac{a}{R}\right)^{mk} \right.\right.$$

$$\left. + \sum_{l=1}^{mk-1} A_l \left(\frac{a}{R}\right)^l \frac{\underline{mk-1}}{\underline{l}\;\underline{mk-l}} \left(\frac{s}{R}\right)^{mk-l} \right] + R\frac{B_{mk}}{mk} \left. \right\} \cos mk\theta$$

$$- \frac{G\tau}{2} R^2 = \text{independent of } \theta.$$

To satisfy this for all values of θ it is necessary that the

coefficient of general term $\cos mk\theta$ must be zero. Consequently for the values of $k=1, 2, 3, \ldots$, it is necessary

$$B_{mk}=-m^2k\frac{a}{R}\left[\frac{A_{mk}}{\underline{mk}}\left(\frac{a}{R}\right)^{mk}+\sum_{l=1}^{mk-1}A_l\left(\frac{a}{R}\right)^{l}\frac{\underline{mk-1}}{\underline{l}\,\underline{mk-l}}\left(\frac{s}{R}\right)^{mk-l}\right]. \quad . \quad . \quad (1)$$

If the assumed coefficients A_n, B_{mn} are determined according to this relation, $\psi_{BA_1A_2\ldots A_p\ldots A_m}-\frac{G\tau}{2}R^2$ takes the constant value of $-\frac{G\tau}{2}R^2$ along the outer boundary.

Along the inner circle the necessary boundary condition is

$$\psi_{A_1A_1A_2\,.\,Ap\ldots AmB}-\frac{G\tau}{2}r_1^2=\psi_{A_1A_1A_2\ldots A_p\ldots A_mB}$$

$$-\frac{G\tau}{2}(a^2+s^2+2as\cos\varphi_1)=\text{independent of }\varphi_1.$$

In this expression the coefficient of general term of $\cos mk\varphi_1$ $(k=1, 2, 3, \ldots)$ is, after some rearrangements,

$$-\frac{G\tau}{2}R\pi\left\{a\left[\frac{A_{mk}}{\underline{mk}}+\frac{(-1)^{mk}}{\underline{mk}}\sum_{l=1}^{\infty}(-1)^lA_l\right.\right.$$

$$\times\sum_{p=2}^{m}\left(\frac{a}{2s\sin\frac{p-1}{m}\pi}\right)^{l+mk}\frac{\underline{l+mk-1}}{\underline{l}}$$

$$\left.\cdot\cos(l-mk)\frac{\pi}{2}\left(1-\frac{2p-1}{m}\right)\right]+R\left[\frac{B_{mk}}{\underline{mk}}\left(\frac{a}{R}\right)^{mk}\right.$$

$$\left.\left.+\left(\frac{a}{R}\right)^{mk}\frac{1}{\underline{mk}}\sum_{l=1}^{\infty}B_{m(k+l)}\frac{\underline{m(k+l)-1}}{\underline{ml}}\left(\frac{s}{R}\right)^{ml}\right]\right\},$$

and that of term of $\cos(mk+q)\varphi_1$ $(q=1, 2, \ldots, (m-1);$ $k=0, 1, 2, 3, \ldots$, with the exception of $k=0$ for $q=1)$ is

$$-\frac{G\tau}{2}R\pi\left\{a\left[\frac{A_{mk+q}}{\underline{mk+q}}+\frac{(-1)^{mk+q}}{\underline{mk+q}}\sum_{l=1}^{\infty}(-1)^lA_l\right.\right.$$

$$\left.\times\sum_{p=2}^{m}\left(\frac{a}{2s\sin\frac{p-1}{m}\pi}\right)^{l+mk+q}\frac{\underline{l+mk+q-1}}{\underline{l}}\right.$$

$$\cdot \cos (l-\overline{mk+q})\frac{\pi}{2}\left(1-\frac{2\overline{p-1}}{m}\right)\Big]$$

$$+\mathrm{R}\left(\frac{a}{\mathrm{R}}\right)^{mk+q}\frac{1}{\underline{|mk+q}}\sum_{l=1}^{\infty}\mathrm{B}_{m(k+l)}\frac{\underline{|m(k+l)-1}}{\underline{|ml-q}}\left(\frac{s}{\mathrm{R}}\right)^{ml-q}\Bigg\},$$

and also that of term of $\cos\varphi_1$ is

$$-\frac{\mathrm{G}\tau}{2}\mathrm{R}\pi a\Big[\mathrm{A}_1-\sum_{l=1}^{\infty}(-1)^l\mathrm{A}_l\sum_{p=2}^{m}\left(\frac{a}{2s\sin\dfrac{p-1}{m}\pi}\right)^{l+1}$$

$$\cdot\cos(l-1)\frac{\pi}{2}\left(1-\frac{2\overline{p-1}}{m}\right)+\sum_{l=1}^{\infty}\mathrm{B}_{ml}\left(\frac{s}{\mathrm{R}}\right)^{ml-1}+\frac{2s}{\pi\mathrm{R}}\Big].$$

To satisfy the necessary condition for all values of φ_1 these three coefficients must be zero. Accordingly the following relations are necessary :—

$$\mathrm{A}_{mk}=-\frac{mk}{a}\Big[a\frac{(-1)^{mk}}{\underline{|mk}}\sum_{l=1}^{\infty}(-1)^l\mathrm{A}_l\sum_{p=2}^{m}\left(\frac{a}{2s\sin\dfrac{p-1}{m}\pi}\right)^{l+mk}$$

$$\cdot\frac{\underline{|l+mk-1}}{\underline{|l}}\cdot\cos(l-mk)\frac{\pi}{2}\left(1-\frac{2\overline{p-1}}{m}\right)+\mathrm{R}\frac{\mathrm{B}_{mk}}{mk}\left(\frac{a}{\mathrm{R}}\right)^{mk}$$

$$+\mathrm{R}\left(\frac{a}{\mathrm{R}}\right)^{mk}\frac{1}{\underline{|mk}}\sum_{l=1}^{\infty}\mathrm{B}_{m(k+l)}\frac{\underline{|m(k+l)-1}}{\underline{|ml}}\left(\frac{s}{\mathrm{R}}\right)^{ml}\Big],$$

(for $k=1, 2, 3, \ldots$), ... (2)

$$\mathrm{A}_{mk+q}=-\frac{mk+q}{a}\Big[a\frac{(-1)^{mk+q}}{\underline{|mk+q}}\sum_{l=1}^{\infty}(-1)^l\mathrm{A}_l$$

$$\times\sum_{p=2}^{m}\left(\frac{a}{2s\sin\dfrac{p-1}{m}\pi}\right)^{l+mk+q}\frac{\underline{|l+mk+q-1}}{\underline{|l}}$$

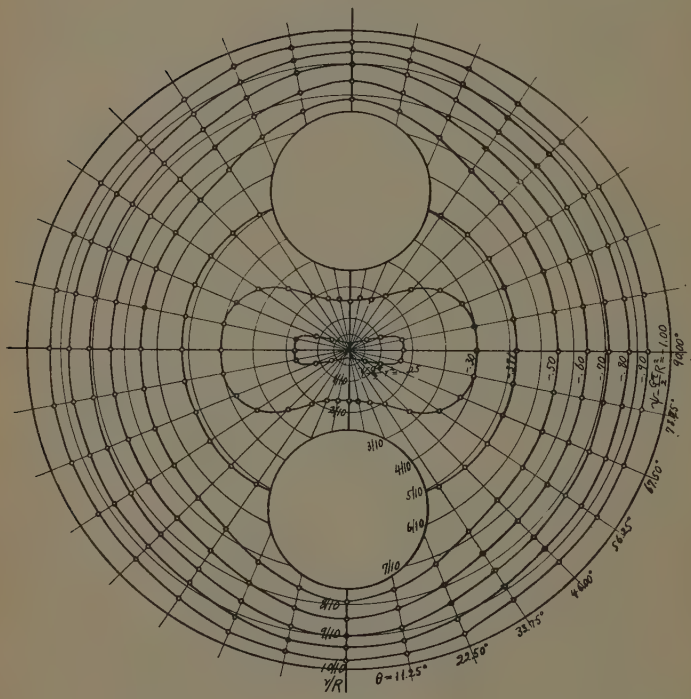
$$\cos(l-\overline{mk+q})\frac{\pi}{2}\left(1-\frac{2\overline{p-1}}{m}\right)$$

$$+\mathrm{R}\left(\frac{a}{\mathrm{R}}\right)^{mk+q}\frac{1}{\underline{|mk+q}}\sum_{l=1}^{\infty}\mathrm{B}_{m(k+l)}\frac{\underline{|m(k+l)-1}}{\underline{|ml-q}}\left(\frac{s}{\mathrm{R}}\right)^{ml-q}\Big],$$

(for $k=0, 1, 2, 3, \ldots$; $q=1, 2\ldots, (m-1)$,
with the exception of $k=0$ for $q=1$), . . (3)

$$A_1 = \frac{1}{1+\sum_{p=2}^{m}\left(\frac{a}{2s\sin\frac{p-1}{m}\pi}\right)^2}\left[\sum_{l=2}^{\infty}(-1)^l A_l\right.$$

Fig. 3.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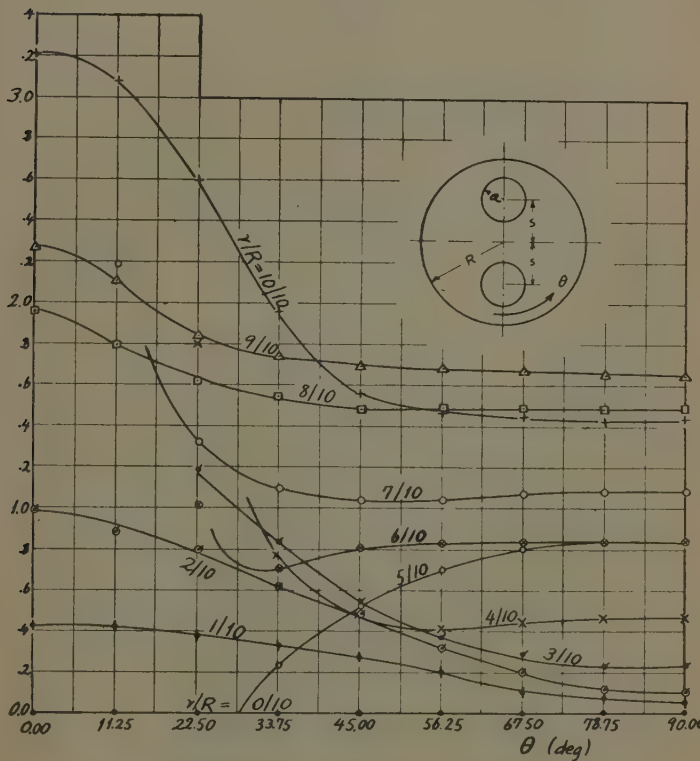
If A_n, B_{mn} are determined to fulfil these relations $\psi_{A_1A_1A_2 \ldots Ap \ldots AmB} - \frac{G\tau}{2} r_1^2$ takes the constant value of

Fig. 4.



Resultant shear stress along sections $r/R = \text{const.}$ $\left(\text{unit} : \frac{G\tau}{2} R\right)$.

(Two circular holes, $s/R = 1/2$, $a/R = 1/4$.)

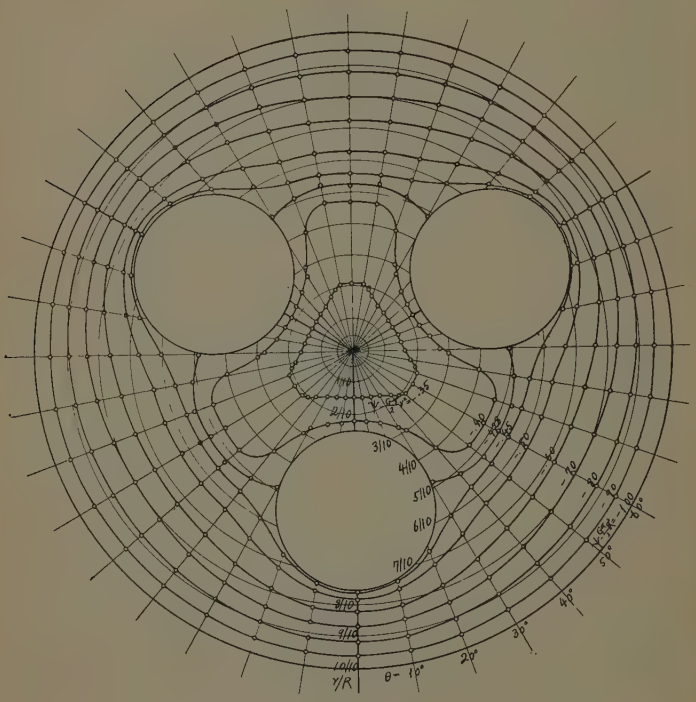
$$-\frac{G\tau}{2} R\pi \left[a \sum_{k=1}^{\infty} (-1)^k \frac{A_k}{k} \sum_{p=2}^{m} \left(\frac{a}{2s \sin \frac{p-1}{m}\pi} \right)^k \right.$$

$$\left. \cdot \cos k\frac{\pi}{2}\left(1 - \frac{2p-1}{m}\right) + R \sum_{\lambda=1}^{m} \frac{B_{mk}}{mk} \left(\frac{s}{R}\right)^{mk} \right] - \frac{G\tau}{2}(a^2+s^2)$$

along the inner boundaries.

The relations (1), (2), (3), and (4) are the simultaneous equations of the first order with infinite number of unknowns A_n, B_{mn}. The method of iteration seems to be one which minimizes the difficulty to get the roots,

Fig. 5.



Lines of shear stress $\left(\text{unit } \psi : \frac{G\tau}{2} R^2\right)$.

(Three circular holes, $s/R = 1/2$, $a/R = 1/4$.)

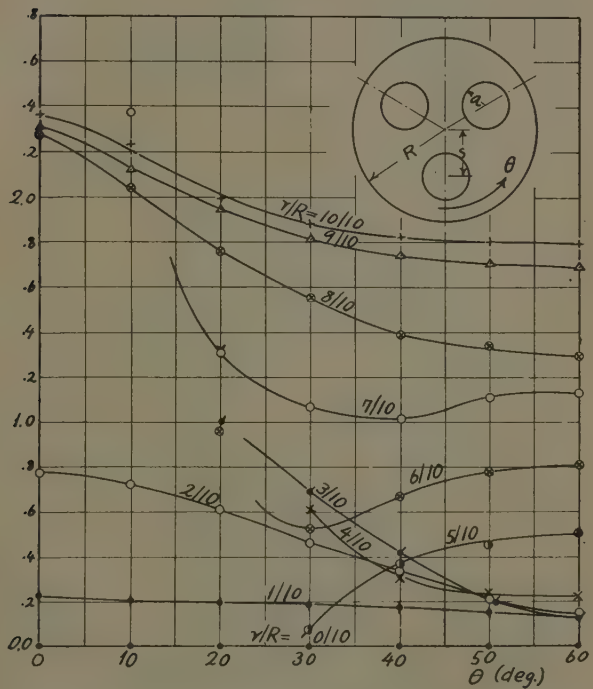
and as for the first approximate values of A_n, B_{mn}, the following seem to be most convenient, *i. e.*,

$$A_1 = -\left(\frac{2s}{\pi R}\right) \Bigg/ \left\{ 1 + \sum_{p=2}^{m} \left[\frac{a}{2s \sin \frac{p-1}{m} \pi} \right]^2 \right\},$$

$$A_{mk} = 0, \quad A_{mk+q} = 0; \quad B_{mk} = 0.$$

Actual examples show that reiteration is sufficient as far as the third.

Fig. 6.



Resultant shear stress along sections $r/R = \text{const.}$ $\left(\text{unit} : \frac{G\tau}{2}R\right)$.

(Three circular holes, $s/R = 1/2$, $a/R = 1/4$.)

III. *Numerical Examples.*

As the numerical examples, the cases having two, three, and four circular holes are calculated. The sections are of the same proportion, namely :

$$s/R = 1/2, \quad a/R = 1/4.$$

One example of the values of assumed coefficients

which were determined by the method described above is as follows (for the case having four circular holes) :—

$$A_1 = -1{\cdot}0241 \times \frac{1}{\pi} \cdot \frac{16}{21}, \qquad B_4 = 0{\cdot}1427 \times \frac{1}{\pi} \cdot \frac{16}{21},$$

$$A_2 = -0{\cdot}1675 \times \text{,, ,} \qquad B_8 = 0{\cdot}0205 \times \text{,, ,}$$

$$A_3 = -0{\cdot}0103 \times \text{,, ,} \qquad B_{12} = 0{\cdot}0021 \times \text{,, ,}$$

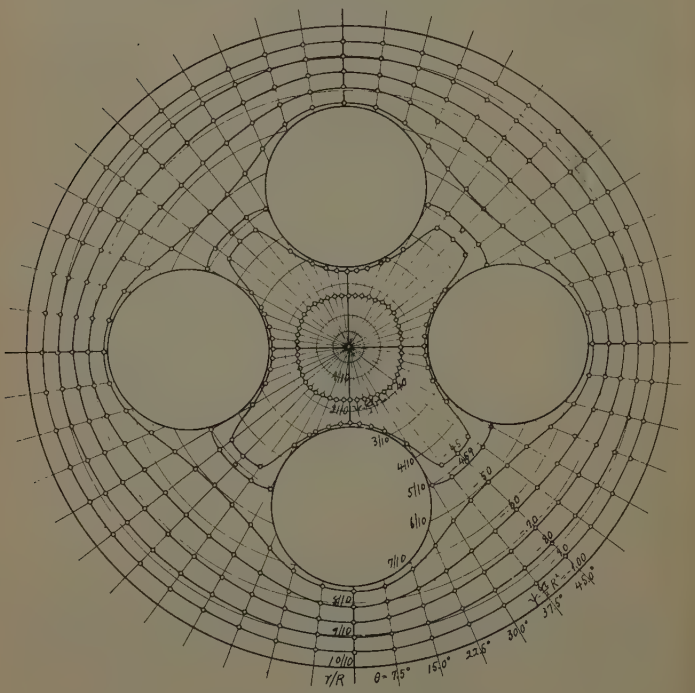
$$A_4 = \ 0{\cdot}0247 \times \text{,, ,} \qquad B_{16} = 0{\cdot}0002 \times \text{,, .}$$

$$A_5 = -0{\cdot}0164 \times \text{,, ,}$$

$$A_6 = \ 0{\cdot}0039 \times \text{,, ,}$$

$$A_7 = \ 0{\cdot}0006 \times \text{,, ;}$$

Fig. 7.

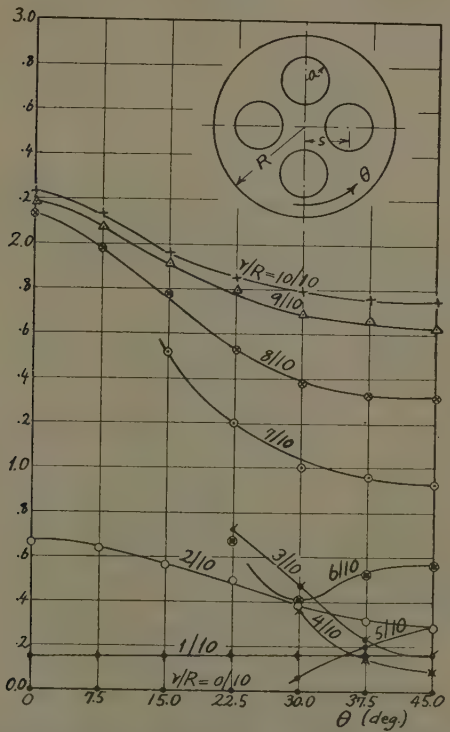


Lines of shear stress $\left(\text{unit } \psi : \frac{G\tau}{2} R^3\right)$.

(Four circular holes, $s/R = 1/2$, $a/R = 1/4$.)

Calculating the values of ψ all over the section by the expressions (I), (I′), (II), and (III), and tracing the curve from one point to another having the equal value of $\psi - \frac{G\tau}{2} r^2$, we get the lines of shear stress as shown in figs. 3, 5, and 7. The slope of $\psi - \frac{G\tau}{2} r^2$ over the section

Fig. 8.



Resultant shear stress along sections $r/R = \text{const.}$ $\left(\text{unit}: \frac{G\tau}{2} R\right)$.

(Four circular holes, $s/R = 1/2$, $a/R = 1/4$.)

at a certain point gives the component shear stress at that point in the direction perpendicular to that in which the slope is measured. The resultant shear stress is given

by the vector sum of the component stresses. It acts tangentially to the lines of shear stress above obtained. Its magnitude is plotted in figs. 4, 6, and 8, taking the radial position of the point as parameter and the angular position as abscissa.

In conclusion, the author expresses his hearty thanks to Prof. S. Yokota, of the Tôkyô Imperial University, for his kind advice in completing this paper.

XCIII. *Fractional Calculus.*

By W. Fabian, *M.A., Ph.D.**

1. *Introduction.*

So far † we have considered properties and applications of the fractional integral, when the lower limit of integration is finite. In the present paper this lower limit will sometimes be infinite.

We define a λth integral of $f(z)$ along a simple curve l by

$$\mathrm{D}^{-\lambda}(l_a)f(z) = \frac{\mathrm{D}^{\gamma}}{\Gamma(\lambda+\gamma)}\int_a^z (z-t)^{\lambda+\gamma-1}f(t)\,dt,$$

where D stands for $\dfrac{d}{dz}$ and γ is the least integer greater than or equal to zero such that $\mathrm{R}(\lambda)+\gamma>0$. When l is a straight line we shall often denote it by j, *and the letter j will not be used in any other sense in the present paper.* The lower limit a of integration, when infinite, will be denoted by $v+ik$, where v and k are real and either v or k may be finite. *The expression $v+ik$ will not be used to denote a finite value of a in the present paper.*

2. *Some Properties of the Fractional Integral.*

(*a*) Theorem 1.—*Let j join points $v+ik$ and c, where c is a point in the finite part of the plane.*

Let $(z-c)^{\lambda+1}f(z)$, where $\mathrm{R}(\lambda)>0$, be finite and continuous on j except possibly at c.

* Communicated by the Author.

† Fabian, Phil. Mag. ser. 7, xx. pp. 781–789 (1935), and xxi. pp. 274–280 (1936); Am. Journ. of Math. and Phys. vol. xv. no. 2 (1936); Math. Gazette, May (1936), and October (1936).

If $(z-c)^{\rho}f(z)\to A$ *as* $z\to c$ *along* j, *where* A *is a constant, and* $R(\rho)\geqslant R(\lambda)+1$, *then*

$$\lim_{z\leftarrow c}.(z-c)^{\rho-\lambda}D^{-\lambda}(j_{v+ik})f(z)=(-1)^{\lambda}\frac{A\Gamma(\rho-\lambda)}{\Gamma(\rho)}.$$

Proof.—$v+ik$ may be written in the form

$$\lim_{u\to 0}.\left(\frac{1}{u}+c\right).$$

Let j be transformed into J by the substitution

$$t'=\frac{1}{t-c},$$

and let

$$z'=\frac{1}{z-c}.$$

J is a straight line joining the origin and a point at infinity. We obtain

$$D^{-\lambda}(j_{v+ik})f(z)=\lim_{u\to 0}.\frac{1}{\Gamma(\lambda)}\int_{\frac{1}{u}+c}^{z}(z-t)^{\lambda-1}f(t)\,dt$$

$$=\lim_{u\to 0}.\frac{1}{\Gamma(\lambda)}\int_{\frac{1}{u}+c}^{z}(z-t)^{\lambda-1}\left\{\frac{F\left(\frac{1}{t-c}\right)}{(t-c)^{\lambda+1}}\right\}dt,$$

where

$$F\left(\frac{1}{t-c}\right)=(t-c)^{\lambda+1}f(t). \quad . \quad . \quad . \quad . \quad (1)$$

Therefore $D^{-\lambda}(j_{v+ik})f(z)$

$$=\lim_{u\to 0}.\frac{(-1)^{\lambda-1}(z-c)^{\lambda-1}}{\Gamma(\lambda)}\int_{\frac{1}{u}+c}^{z}\left(\frac{1}{z-c}-\frac{1}{t-c}\right)^{\lambda-1}$$

$$\times F\left(\frac{1}{t-c}\right)\frac{dt}{(t-c)^2}$$

$$=\lim_{u\to 0}.\frac{(-1)^{\lambda}(z-c)^{\lambda-1}}{\Gamma(\lambda)}\int_{u}^{z'}(z'-t')^{\lambda-1}F(t')\,dt'$$

$$=(-1)^{\lambda}(z-c)^{\lambda-1}\left(\frac{d}{dz'}\right)^{-\lambda}(J_0)F(z'). \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

By (1) and the hypothesis, $F(z')$ is finite and continuous on J except possibly at infinity, and

$$\frac{F(z')}{(z')^{\rho-\lambda-1}}\to A \quad \text{as } |z'|\to\infty \text{ along J}.$$

Therefore, by a previous theorem *,

$$\lim_{|z'|\to\infty} \frac{\left(\frac{d}{dz'}\right)^{-\lambda}(\mathrm{J}_0)\mathrm{F}(z')}{(z')^{\rho-1}} = \frac{\mathrm{A}\Gamma(\rho-\lambda)}{\Gamma(\rho)},$$

so that, by (2),

$$\lim_{z\to c} \frac{\mathrm{D}^{-\lambda}(j_{v+ik})f(z)}{(-1)^{\lambda}(z-c)^{\lambda-\rho}} = \frac{\mathrm{A}\Gamma(\rho-\lambda)}{\Gamma(\rho)}.$$

The theorem is proved.

(*b*) Denote the following conditions by ϕ :—

$f(z)$ can be expanded in the half-plane $x>x_0$ in the Dirichlet series

$$\sum_{n=1}^{\infty} g_n e^{-\sigma_n z},$$

whose line of convergence is $x=x_0$, where g_n is bounded and independent of z, and σ_n is a sequence of real increasing numbers whose limit is infinity, σ_1 being positive. a, the lower limit of integration for $\mathrm{D}^{-\lambda}(l_a)f(z)$, is at infinity in the half-plane $x>x_0$, and is such that v is infinite and k finite. We shall denote the half-plane $x>x_0$ by H.

Theorem 2.—*If l lies wholly in* H, *and conditions ϕ are satisfied, then* $\mathrm{D}^{-\lambda}(l_a)f(z)$ *exists and is analytic in* H, *and* $\mathrm{D}^{-\lambda}(l_a)f(z)$ *can be expanded in* H *in the Dirichlet series*

$$\sum_{n=1}^{\infty} g_n(-\sigma_n)^{-\lambda}e^{-\sigma_n z},$$

which has $x=x_0$ for its line of convergence. Also, in H,

$$\mathrm{D}^{-\lambda}(l_a)\mathrm{D}^{-\mu}(l_a)f(z) = \mathrm{D}^{-\mu}(l_a)\mathrm{D}^{-\lambda}(l_a)f(z) = \mathrm{D}^{-\lambda-\mu}(l_a)f(z).$$

Proof.—Let us first suppose that

$$(z-t)^{\lambda+\gamma-1} = e^{(\lambda+\gamma-1)\log(z-t)}.$$

Consider the t-plane, in which z is momentarily considered a fixed point in H. Let a line I be drawn in H parallel to the real axis, starting from z and extending to infinity to the right. Let τ be a point on I to the right of z.

* Fabian, Am. Journ. of Math. and Phys. xv. no. 2 (1936).

If we put $g_n=P_n+iQ_n$, $t=p+iq$, $\lambda=A+iB$, where P_n, Q_n, p, q, A, and B are real, then the series

$$S(t)=\sum_{n=1}^{\infty} g_n(z-t)^{\lambda+\gamma-1}e^{-\sigma_n t}$$

can be written

$$\begin{aligned}S(t)=\sum_{n=1}^{\infty}(z-t)^{A+\gamma-1}e^{-\sigma_n p}&[P_n\cos\{B\log(z-t)-q\sigma_n\}\\&+Q_n\sin\{q\sigma_n-B\log(z-t)\}+iP_n\sin\{B\log(z-t)-q\sigma_n\}\\&+iQ_n\cos\{q\sigma_n-B\log(z-t)\}]\\&=S_1(t)+iS_2(t),\end{aligned}$$

where $S_1(t)$ and $S_2(t)$ consist of real terms for all values of t on I.

It can be easily seen that sufficient conditions * are satisfied for the term-by-term integration of $S_1(t)$ and $S_2(t)$, and hence of $S(t)$, over the intervals $(z,\ \tau)$ and $(\tau,\ \infty+iy)$ on I. Hence, integrating along I,

$$\int_{\tau}^{z}S(t)\,dt=\sum_{n=1}^{\infty}g_n\int_{\tau}^{z}(z-t)^{\lambda+\gamma-1}e^{-\sigma_n t}\,dt,$$

$$\int_{\infty+iy}^{\tau}S(t)\,dt=\sum_{n=1}^{\infty}g_n\int_{\infty+iy}^{\tau}(z-t)^{\lambda+\gamma-1}e^{-\sigma_n t}\,dt,$$

so that

$$\int_{\infty+iy}^{z}S(t)\,dt=\sum_{n=1}^{\infty}g_n\int_{\infty+iy}^{z}(z-t)^{\lambda+\gamma-1}e^{-\sigma_n t}\,dt. \quad . \quad (1)$$

Putting $t=z+t'$, we get

$$\begin{aligned}\int_{\infty+iy}^{z}(z-t)^{\lambda+\gamma-1}e^{-\sigma_n t}\,dt&=(-1)^{\lambda+\gamma-1}e^{\sigma_n z}\int_{\infty}^{0}(t')^{\lambda+\gamma-1}e^{-\sigma_n t'}\,dt'\\&=\frac{e^{-\sigma_n z}\Gamma(\lambda+\gamma)}{(-\sigma_n)^{\lambda+\gamma}}.\end{aligned}$$

Hence, by (1),

$$\frac{1}{\Gamma(\lambda+\gamma)}\int_{\infty+iy}^{z}S(t)\,dt=\sum_{n=1}^{\infty}g_n(-\sigma_n)^{-\lambda-\gamma}e^{-\sigma_n z},$$

that is,

$$D^{-\lambda-\gamma}(I_{\infty+iy})f(z)=\sum_{n=1}^{\infty}g_n(-\sigma_n)^{-\lambda-\gamma}e^{-\sigma_n z}. \quad . \quad . \quad (2)$$

* Carslaw, 'Fourier's Series and Integrals' (1930), pp. 173 and 178.

Multiplying (2) by $e^{2N\pi i\lambda}$, where N is any integer, we get the relation for the general values of $(z-t)^{\lambda+\gamma-1}$ and (consequently) of $(-\sigma_n)^{-\lambda-\gamma}$.

To show that

$$\mathrm{D}^{-\lambda-\gamma}(\mathrm{I}_{\infty+ty})f(z) = \mathrm{D}^{-\lambda-\gamma}(l_a)f(z), \quad . \quad . \quad . \quad (3)$$

we form a closed contour zz_1z_2z, consisting of a part zz_1 of I, of a line z_1z_2 drawn through z_1 perpendicular to I, and of the arc z_2z cut off on l by the line z_1z_2.

Equality (3) then follows from the relation

$$\int_{zz_1}(z-t)^{\lambda+\gamma-1}f(t)\,dt = \left(\int_{zz_2}+\int_{z_2z_1}\right)(z-t)^{\lambda+\gamma-1}ft\,dt$$

on letting z_1 tend to infinity along I. Hence, by (2) and (3),

$$\mathrm{D}^{-\lambda}(l_a)f(z) \equiv \mathrm{D}^{\gamma}\mathrm{D}^{-\lambda-\gamma}(l_a)f(z)$$

$$= \sum_{n=1}^{\infty} g_n(-\sigma_n)^{-\lambda}e^{-\sigma_n z},$$

and by the theory of Dirichlet series $x=x_0$ is the line of convergence of this series.

By the same argument by which we obtained the Dirichlet series for $\mathrm{D}^{-\lambda}(l_a)f(z)$ we get, in H,

$$\mathrm{D}^{-\mu}(l_a)\mathrm{D}^{-\lambda}(l_a)f(z) = \sum_{n=1}^{\infty} g_n(-\sigma_n)^{-\lambda}(-\sigma_n)^{-\mu}e^{-\sigma_n z}$$

$$= \sum_{n=1}^{\infty} g_n(-\sigma_n)^{-\lambda-\mu}e^{-\sigma_n z}$$

$$= \mathrm{D}^{-\lambda-\mu}(l_a)f(z).$$

Similarly,

$$\mathrm{D}^{-\lambda}(l_a)\mathrm{D}^{-\mu}(l_a)f(z) = \mathrm{D}^{-\lambda-\mu}(l_a)f(z).$$

The proof of the theorem is completed.

THEOREM 3.—*If $f(z)$ is analytic on l, and conditions ϕ are satisfied, then each branch of $\mathrm{D}^{-\lambda}(l_a)f(z)$ exists and is analytic throughout the finite region of analytic continuity of $f(z)$. Also, in this region,*

$$\mathrm{D}^{-\lambda}(l_a)\mathrm{D}^{-\mu}(l_a)f(z) = \mathrm{D}^{-\mu}(l_a)\mathrm{D}^{-\lambda}(l_a)f(z)$$

$$= \mathrm{D}^{-\lambda-\mu}(l_a)f(z).$$

Proof.—Let a point w divide l into two parts, l_1 and l_2, such that l_1 lies wholly in H and does not contain the upper limit of integration for $\mathrm{D}^{-\lambda}(l_a)f(z)$. We have

$$\mathrm{D}^{-\lambda-\gamma}(l_a)f(z) = \mathrm{D}^{-\lambda-\gamma}(l_w)f(z) + \frac{1}{\Gamma(\lambda+\gamma)}\int_{l_1}(z-t)^{\lambda+\gamma-1}f(t)\,dt, \quad (1)$$

from which it can be seen, since the latter integral converges, that $\mathrm{D}^{-\lambda-\gamma}(l_a)f(z)$ exists in the finite region of analytic continuity of $f(z)$. We find, on putting the Dirichlet series for $f(t)$, that the integral

$$\int_{l_1}(z-t)^{\lambda+\gamma-m}f(t)\,dt$$

converges uniformly with regard to z for every integral value of m greater than and equal to unity. Therefore the differential coefficients

$$\left(\frac{d}{dz}\right)^m\int_{l_1}(z-t)^{\lambda+\gamma-1}f(t)\,dt \equiv \prod_{n=1}^{m}(\lambda+\gamma-n)\int_{l_1}(z-t)^{\lambda+\gamma-m-1}f(t)\,dt$$

exist, so that each branch of

$$\int_{l_1}(z-t)^{\lambda+\gamma-1}f(t)\,dt$$

is analytic in the region considered, points on l_1 being excluded. Also *, each branch of $\mathrm{D}^{-\lambda-\gamma}(l_w)f(z)$ is analytic in the finite region of analytic continuity of $f(z)$, $z=w$ being excluded. Consequently, by (1), each branch of $\mathrm{D}^{-\lambda-\gamma}(l_a)f(z)$, and hence of $\mathrm{D}^{-\lambda}(l_a)f(z)$, is analytic throughout the finite region of analytic continuity of $f(z)$.

Similarly, each branch of $\mathrm{D}^{-\lambda-\mu}(l_a)f(z)$ and of $\mathrm{D}^{-\mu}(l_a)\mathrm{D}^{-\lambda}(l_a)f(z)$ is analytic throughout this region. And since, by Theorem 2,

$$\mathrm{D}^{-\mu}(l_a)\mathrm{D}^{-\lambda}(l_a)f(z) = \mathrm{D}^{-\lambda-\mu}(l_a)f(z),$$

when l lies wholly in H, this relation holds, by the principle of analytic continuation, throughout the finite region of analytic continuity of $f(z)$. Similarly, throughout this region,

$$\mathrm{D}^{-\lambda}(l_a)\mathrm{D}^{-\mu}(l_a)f(z) = \mathrm{D}^{-\lambda-\mu}(l_a)f(z).$$

* Fabian, Math. Gazette (1936).

THEOREM 4.—*Let j join $v+ik$ and c, where c is a point in the finite part of the plane. Let conditions ϕ, when l is j, be satisfied. Let $f(z)$ be analytic on j except possibly at c.*

If

$$\lim_{z\to c}(z-c)^{\rho}\mathrm{D}^{\lambda}(j_{v+ik})f(z)=\mathrm{A},$$

a constant, where $\mathrm{R}(\lambda)>0$, $\mathrm{R}(\rho)\geqslant\mathrm{R}(\lambda)+1$, *then*

$$(z-c)^{\rho-\lambda}f(z)\to(-1)^{\lambda}\frac{\mathrm{A}\Gamma(\rho-\lambda)}{\Gamma(\rho)}$$

as $z\to c$ along j.

Proof.—By Theorem 2 we have, in the half-plane $x>x_0$,

$$\mathrm{D}^{\lambda}(j_{v+ik})f(z)=\sum_{n=1}^{\infty}g_n(-\sigma_n)^{\lambda}e^{-\sigma_n z}.$$

Therefore $(z-c)^{\lambda+1}\mathrm{D}^{\lambda}(j_{v+ik})f(z)$ is finite at infinity on j. By Theorem 3 each branch of $\mathrm{D}^{\lambda}(j_{v+ik})f(z)$, and hence of $(z-c)^{\lambda+1}\mathrm{D}^{\lambda}(j_{v+ik})f(z)$, is analytic on the finite part of j except possibly at c. Therefore

$$(z-c)^{\lambda+1}\mathrm{D}^{\lambda}(j_{v+ik})f(z)$$

is finite and continuous on j except possibly at c. Hence, by Theorem 1,

$$\lim_{z\to c}(z-c)^{\rho-\lambda}\mathrm{D}^{-\lambda}(j_{v+ik})\mathrm{D}^{\lambda}(j_{v+ik})f(z)=\frac{(-1)^{\lambda}\mathrm{A}\Gamma(\rho-\lambda)}{\Gamma(\rho)}.$$

By Theorem 3

$$\mathrm{D}^{-\lambda}(j_{v+ik})\mathrm{D}^{\lambda}(j_{v+ik})f(z)=f(z)$$

in the finite region of analytic continuity of $f(z)$. Hence the conclusion.

THEOREM 5.—*Let j join $v+ik$ and c, where c is a point in the finite part of the plane, and let b be a fixed point on j between $v+ik$ and c. Let conditions ϕ, when l is j, be satisfied. Let $f(z)$ be analytic on j except possibly at c.*

If

$$\lim_{z\to c}(z-c)^{\rho}\mathrm{D}^{\lambda}(j_b)f(z)=\mathrm{A},$$

a constant, where $\mathrm{R}(\lambda)>0$, $\mathrm{R}(\rho)\geqslant\mathrm{R}(\lambda)+1$, *then*

$$(z-c)^{\rho-\lambda}f(z)\to(-1)^{\lambda}\frac{\mathrm{A}\Gamma(\rho-\lambda)}{\Gamma(\rho)}$$

as $z\to c$ along j.

Proof.—On j, in H, take a point w different from c, dividing j into two parts. Then

$$\mathrm{D}^{\lambda}(j_b)f(z) = \mathrm{D}^{\lambda}(j_{v+ik})f(z) - \frac{\mathrm{D}^{\delta}}{\Gamma(\delta-\lambda)}\int_w^b (z-t)^{\delta-\lambda-1}f(t)\,dt - \frac{\mathrm{D}^{\delta}}{\Gamma(\delta-\lambda)}\int_{v+ik}^w (z-t)^{\delta-\lambda-1}f(t)\,dt, \quad (1)$$

where δ is the least integer greater than or equal to zero such that $\mathrm{R}(-\lambda)+\delta>0$.

The differential coefficients of

$$\int_w^b (z-t)^{\delta-\lambda-1}f(t)\,dt \quad \text{at } z=c,$$

obtained by differentiation with respect to z under the integral sign *, all exist, so that each branch of

$$\frac{\mathrm{D}^{\delta}}{\Gamma(\delta-\lambda)}\int_w^b (z-t)^{\delta-\lambda-1}f(t)\,dt$$

is analytic at c. Consequently

$$\lim_{z\to c}(z-c)^{\rho}\cdot\frac{\mathrm{D}^{\delta}}{\Gamma(\delta-\lambda)}\int_w^b (z-t)^{\delta-\lambda-1}f(t)\,dt = 0.$$

Again, the integral

$$\int_{v+ik}^w (z-t)^{\delta-\lambda-1}f(t)\,dt,$$

where $f(t)$ can be replaced by its expansion in the Dirichlet series, can be differentiated at c with respect to z under the integral sign as many times as we please †, so that each branch of

$$\frac{\mathrm{D}^{\delta}}{\Gamma(\delta-\lambda)}\int_{v+ik}^w (z-t)^{\delta-\lambda-1}f(t)\,dt$$

is analytic at c. Hence

$$\lim_{z\to c}(z-c)^{\rho}\cdot\frac{\mathrm{D}^{\delta}}{\Gamma(\delta-\lambda)}\int_{v+ik}^{w'} (z-t)^{\delta-\lambda-1}f(t)\,dt = 0.$$

* This is justified by arguments of classical mathematics.
† This follows from the arguments of classical mathematics.

Therefore, by (1),

$$\lim_{z\to c}(z-c)^{\rho}\mathrm{D}^{\lambda}(j_{v+ik})f(z)=\lim_{z\to c}(z-c)^{\rho}\mathrm{D}^{\lambda}(j_b)f(z)$$
$$=\mathrm{A},\quad\text{by hypothesis.}$$

The conclusion follows now from Theorem 4.

THEOREM 6.—*Let j join $v+ik$ and c, where c is a point in the finite part of the plane, and let b be a fixed point on j between $v+ik$ and c. Let conditions ϕ, when l is j, be satisfied. Let $f(z)$ be finite and continuous on j except possibly at c.*

If $(z-c)^{\rho}f(z)\to\mathrm{A}$, a constant, as $z\to c$ along j, where $\mathrm{R}(\rho)\geqslant\mathrm{R}(\lambda)+1$, $\mathrm{R}(\lambda)>0$, then

$$\lim_{z\to c}(z-c)^{\rho-\lambda}\mathrm{D}^{-\lambda}(j_b)f(z)=(-1)^{\lambda}\frac{\mathrm{A}\Gamma(\rho-\lambda)}{\Gamma(\rho)}.$$

Proof.—Under the conditions of the hypothesis $(z-c)^{\lambda+1}f(z)$ is finite and continuous on j except possibly at c. Therefore, by Theorem 1,

$$\lim_{z\to c}(z-c)^{\rho-\lambda}\mathrm{D}^{-\lambda}(j_{v+ik})f(z)=\frac{(-1)^{\lambda}\mathrm{A}\Gamma(\rho-\lambda)}{\Gamma(\rho)}. \qquad (1)$$

Taking a point w on j in H, different from c, so as to divide j into two parts, we have

$$\mathrm{D}^{-\lambda}(j_{v+ik})f(z)=\mathrm{D}^{-\lambda}(j_b)f(z)$$
$$+\frac{\mathrm{D}^{\gamma}}{\Gamma(\lambda+\gamma)}\int_w^b(z-t)^{\lambda+\gamma-1}f(t)\,dt$$
$$+\frac{\mathrm{D}^{\gamma}}{\Gamma(\lambda+\gamma)}\int_{v+ik}^w(z-t)^{\lambda+\gamma-1}f(t)\,dt. \qquad (2)$$

Each of the branches of the latter two integrals is analytic at c*, so that

$$\lim_{z\to c}(z-c)^{\rho-\lambda}\cdot\frac{\mathrm{D}^{\gamma}}{\Gamma(\lambda+\gamma)}\int_w^b(z-t)^{\lambda+\gamma-1}f(t)\,dt=0, \qquad (3)$$

$$\lim_{z\to c}(z-c)^{\rho-\lambda}\cdot\frac{\mathrm{D}^{\gamma}}{\Gamma(\lambda+\gamma)}\int_{v+ik}^w(z-t)^{\lambda+\gamma-1}f(t)\,dt=0. \qquad (4)$$

By (1), (2), (3), and (4) the conclusion follows.

* *Loc. cit.*

XCIV. *Lebesgue Complex Integration.*

By W. Fabian, *M.A., Ph.D.**

1. *Introduction.*

CONTINUING our work on the extension of the theory of Riemann integration of functions of a complex variable, which we began in a previous paper †, we shall discuss now the absolute integrability of such functions, a property which we shall define here. Its relation with the absolute convergence of Lebesgue integrals will be pointed put. The continuity of the complex Lebesgue integral will also be discussed.

2. *Absolute Integrability.*

(*a*) Let l be a bounded simple curve, defined by

$$z=x+iy=x(t)+iy(t)$$

in the finite interval $(t_0 \leqslant t \leqslant \mathrm{T})$, such that $x(t)$ and $y(t)$ are monotonic in this interval. Let $\mathrm{E}(l)$ be a set of points on l, and E_t the corresponding set on the t-segment. The Lebesgue integral of a function $f(z)=\mathrm{P}+i\mathrm{Q}$, where P and Q are real, taken over $\mathrm{E}(l)$, is defined by ‡

$$\int_{\mathrm{E}(l)} f(z)\,dz=\int_{\mathrm{E}_t}\mathrm{P}\,dx\,(t)-\int_{\mathrm{E}_t}\mathrm{Q}\,dy\,(t)$$
$$+i\int_{\mathrm{E}_t}\mathrm{Q}\,dx\,(t)+i\int_{\mathrm{E}_t}\mathrm{P}\,dy\,(t),$$

if each of these Lebesgue-Stieltjes integrals, taken over E_t, exists.

Now let E_s be the set of points corresponding to $\mathrm{E}(l)$ on the s-segment $(0 \leqslant s \leqslant \mathrm{S})$, where s denotes the length of that arc of l which corresponds to the interval (t_0, t), and $s=\mathrm{S}$ when $t=\mathrm{T}$.

If the Lebesgue integral $\int_{\mathrm{E}_s} |f(z)|\,ds$, taken over E_s,

* Communicated by the Author.

† Fabian, Phil. Mag. (Sept. 1936). Generalized differentiation (the inverse operation of Lebesgue integration) is also investigated here.

‡ Fabian, Phil. Mag. (Sept. 1936). As pointed out here, this theory can be immediately extended to the case of integration on curves which consist of a finite number of such bounded simple curves l.

exists as a finite number, then we shall say that $f(z)$ is absolutely integrable over $\mathrm{E}(l)$ in the sense of Lebesgue, and we shall write

$$\int_{\mathrm{E}_s} |f(z)|\,ds = \int_{\mathrm{E}(l)} |f(z)|\,|dz|.$$

If both the Lebesgue integrals

$$\int_{\mathrm{E}(l)} f(z)\,dz \quad \text{and} \quad \int_{\mathrm{E}(l)} |f(z)|\,|dz|$$

exist as finite numbers, then we shall say that the Lebesgue integral $\int_{\mathrm{E}(l)} f(z)\,dz$ converges absolutely *.

$\mathrm{E}(l)$ is said to be measurable if the corresponding sets of points E_x and E_y, on the x- and y-segments respectively, are measurable ; and $f(z)$ is said to be measurable in $\mathrm{E}(l)$ if P and Q are measurable in E_x and E_y.

(*b*) We shall establish now three theorems on absolute integrability.

Theorem 1.—*If $f(z)$ is measurable in, and integrable over, a measurable set of points $E(l)$, then $f(z)$ is absolutely integrable over $E(l)$.*

Proof.—By a previous theorem †,

$$\int_{\mathrm{E}(l)} f(z)\,dz = \int_{\mathrm{E}_s} f(z)\frac{dz}{ds}\,ds.$$

Since $\mathrm{E}(l)$ is measurable, E_s is so †, and since $f(z)$ is measurable in $\mathrm{E}(l)$, therefore † $f(z)$, and hence $f(z)\dfrac{dz}{ds}$ is measurable in E_s. Consequently, also $\left|f(z)\dfrac{dz}{ds}\right|$ is measurable in E_s. Putting

$$f(z)\frac{dz}{ds} = \phi_1(s) + i\phi_2(s),$$

* Since $f(z)$ need not be measurable in $\mathrm{E}(l)$ if $|f(z)|$ is measurable in E_s, the existence of $\int_{\mathrm{E}(l)} |f(z)|\,|dz|$ does not imply the existence of $\int_{\mathrm{E}(l)} f(z)\,dz$. For a discussion of measure and measurable functions see our previous paper.

† Fabian, Phil. Mag. (Sept. 1936).

where $\phi_1(s)$ and $\phi_2(s)$ are real, we have

$$\int_{E_s} \left| f(z) \frac{dz}{ds} \right| ds \leqslant \int_{E_s} \left| \phi_1(s) \right| ds + \int_{E_s} \left| \phi_2(s) \right| ds,$$

so that the integral

$$\int_{E_s} \left| f(z) \frac{dz}{ds} \right| ds \equiv \int_{E_s} \left| f(z) \right| ds$$

exists. The theorem is proved.

Theorem 2.—*If* $f(z)$ *is measurable in, and absolutely integrable over, a measurable set of points* $E(l)$, *then* $f(z)$ *is integrable over* $E(l)$.

Proof.—Since $E(l)$ is measurable, E_s is so *, and $f(z)$, being measurable in $E(l)$, is therefore * measurable in E_s. Hence also $| f(z) |$ is measurable in $E(l)$ and E_s.

Let $[f(z)]_N$ be a function, which is equal to $[f(z)]$ when $[f(z)] \leqslant N$, and which is equal to N when $[f(z)] > N$. Divide the interval $\{0 \leqslant [f(z)]_N \leqslant N\}$ into p parts (a_{r-1}, a_r), for $r=1, 2, 3, \ldots p$, where $a_0=0$ and $a_p=N$. Let $g_r(s)$ and $g_r(x)$ be the sets of points in E_s and E_x respectively, for which

$$a_{r-1} \leqslant |f(z)|_N < a_r \qquad \text{for } r=1, 2, 3, \ldots p-1,$$

and $$a_{r-1} \leqslant |f(z)|_N \leqslant a_r \qquad \text{for } r=p.$$

Let ${}_r\triangle_n(s)$ denote a set of non-overlapping intervals on the s-segment, and ${}_r\triangle_n(x)$ the corresponding set of intervals on the x-segment, where ${}_r\triangle_n(s)$ is such that, for a given r, the points of $g_r(s)$ are common to all the sets of a sequence of sets of intervals ${}_r\triangle_n(s)$, each of which sets contains the next. Then ${}_r\triangle_n(x)$ is a set of non-overlapping intervals, and the points of $g_r(x)$ are common, for a given r, to all the sets of the corresponding sequence of sets of intervals ${}_r\triangle_n(x)$, each of which sets also contains the next.

We have

$$m[g_r(s)] = \lim_{n \to \infty} \mathrm{M}[{}_r\Delta_n(s)],$$

$$m[g_r(x)] = \lim_{n \to \infty} \mathrm{M}[{}_r\Delta_n(x)],$$

where $m[g_r(s)]$ and $m[g_r(x)]$ are the measures of $g_r(s)$ and $g_r(x)$, and $\mathrm{M}[{}_r\triangle_n(s)]$ and $\mathrm{M}[{}_r\triangle_n(x)]$ the sums of the lengths of the intervals of ${}_r\triangle_n(s)$ and of the intervals of ${}_r\triangle_n(x)$ respectively.

* *Loc. cit.*

Now

$$\mathrm{M}[{}_{r}\Delta_{n}(s)] \geqslant \mathrm{M}[{}_{r}\Delta_{n}(x)].$$

Therefore $m[g_r(s)] \geqslant m[g_r(x)]$.

Consequently

$$\sum_{r=1}^{p} a_{r-1} m[g_r(s)] \geqslant \sum_{r=1}^{p} a_{r-1} m[g\,(x)].$$

Hence $$\int_{\mathrm{E}_s} |f(z)|\,ds \geqslant \int_{\mathrm{E}_x} |f(z)|\,dx,$$

so that the integral

$$\int_{\mathrm{E}_x} |f(z)|\,dx$$

exists. And since

$$\int_{\mathrm{E}_x} |f(z)|\,dx \geqslant \int_{\mathrm{E}_x} |\mathrm{P}|\,dx$$

and $$\int_{\mathrm{E}_x} |f(z)|\,dx \geqslant \int_{\mathrm{E}_x} |\mathrm{Q}|\,dx,$$

it follows that P and Q are integrable over E_x. Similarly P and Q are integrable over E_y. Hence the conclusion.

Theorem 3.—*If the integral of* $f(z)$ *over a measurable set of points* $E(l)$ *converges absolutely, then*

$$\left| \int_{\mathrm{E}(l)} f(z)\,dz \right| \leqslant \int_{\mathrm{E}(l)} |f(z)|\,|dz|.$$

Proof.—The theorem will be proved if we show that *

$$\left| \int_{\mathrm{E}_s} f(z) \frac{dz}{ds}\,ds \right| \leqslant \int_{\mathrm{E}_s} \left| f(z) \frac{dz}{ds} \right| ds,$$

E_s being a measurable set.

To show this, let

$$\chi(s) = \phi_1(s) + i\phi_2(s),$$

where $\phi_1(s)$ and $\phi_2(s)$ are real, be a function such that

$$\chi(s) = f(z) \frac{dz}{ds} \quad \text{at all points of } \mathrm{E}_s,$$

and $\chi(s) = 0$ at all points of the complement of E_s relatively to the interval $(0 \leqslant s \leqslant \mathrm{S})$. A net can be fitted on to

* This is known to be true when $f(z) \frac{dz}{ds}$ is real.

$(0 \leqslant s \leqslant S)$, and a set of points $p_1, p_2, \ldots . p_n$ can be determined in the n meshes δ_r of the net, such that, for an arbitrarily chosen positive number ϵ,

$$\left| \int_0^S \phi_1(s)\, ds - \sum_{r=1}^{n} \phi_1(p_r) m(\delta_r) \right| < \epsilon,$$

$$\left| \int_0^S \phi_2(s)\, ds - \sum_{r=1}^{n} \phi_2(p_r) m(\delta_r) \right| < \epsilon,$$

$$\left| \int_0^S |\chi(s)|\, ds - \sum_{r=1}^{n} |\chi(p_r)|\, m(\delta_r) \right| < \epsilon,$$

where $m(\delta_r)$ denotes the length of the mesh δ_r*. Now

$$\left| \sum_{r=1}^{n} \phi_1(p_r) m(\delta_r) + i \sum_{r=1}^{n} \phi_2(p_r) m(\delta_r) \right| \leqslant \sum_{r=1}^{n} |\chi(p_r)|\, m(\delta_r),$$

so that

$$\left| \int_{E_s} \chi(s)\, ds \right| \leqslant \int_{E_s} |\chi(s)|\, ds.$$

The theorem is proved.

(*c*) The theory of absolute integrability enables us to prove the converse of the following theorem, which we have established in our previous paper † :—

Theorem 4.—*If the measurable function* $f(z)$ *is integrable over a measurable set of points* $E(l)$, *then*

$$\int_{E(l)} f(z)\, dz = \int_{E_s} f(z) \frac{dz}{ds} ds.$$

The converse to this theorem is :

Theorem 5.—*If the integral* $\int_{E_s} f(z) \frac{dz}{ds} ds$, *where* $f(z)$ *is measurable in the measurable set of points* E_s, *exists as a finite number, then*

$$\int_{E_s} f(z) \frac{dz}{ds} ds = \int_{E(l)} f(z)\, dz.$$

Proof.—Since E_s is measurable, therefore $E(l)$ is so, and $f(z)$, being measurable in E_s, is also measurable in $E(l)$ ‡.

* Hobson, ' Theory of Functions of a Real Variable,' vol. i. p. 642 (1927).

† *Loc. cit.* ‡ *Loc. cit.*

By theorem 1 and the hypothesis the integral

$$\int_{E(l)} |f(z)|\,|dz| \equiv \int_{E_s} \left| f(z)\frac{dz}{ds} \right| ds$$

exists. Hence, by theorem 2, the integral $\int_{E(l)} f(z)dz$ exists. Therefore, by theorem 4,

$$\int_{E(l)} f(z)\,dz = \int_{E_s} f(z)\frac{dz}{ds}\,ds.$$

3. *Continuity of the Complex Lebesgue Integral.*

If $$\phi\{E(l)\} = \int_{E(l)} f(z)\,dz$$

exists for all measurable sets of points $E(l)$ in a fixed measurable set of points $G(l)$, then $\phi\{E(l)\}$ is called the generalized indefinite integral of $f(z)$ over $E(l)$.

Theorem 6.—*The generalized indefinite integral* $\int_{E(l)} f(z)dz$ *of the measurable function* $f(z)$, *taken over the variable measurable set of points* $E(l)$ *in a fixed measurable set of points* $G(l)$, *is a continuous function of* $E(l)$ *for variations of* $E(l)$ *in* $G(l)$.

Proof.—Let G_s be the set of points on the s-segment $(0 \leqslant s \leqslant S)$, which corresponds to $G(l)$. To every measurable set of points $E(l)$ in $G(l)$ there corresponds a measurable set of points E_s in G_s, and conversely *. By theorems 4 and 5 both the integrals

$$\int_{E(l)} f(z)\,dz \quad \text{and} \quad \int_{E_s} f(z)\frac{dz}{ds}\,ds$$

exist and are equal to each other if either of these integrals exists. Hence, by the hypothesis, these integrals exist and are equal to each other for all the measurable sets $E(l)$ in $G(l)$ and E_s in G_s respectively. Now, the generalized indefinite integral $\int_{E_s} f(z)\frac{dz}{ds}\,ds$ is a continuous function † of E_s for variations of the measurable set E_s in G_s. The conclusion follows.

* *Loc. cit.*

† Hobson, 'Theory of Functions of a Real Variable,' vol. i. p. 559 (1921).

XCV. *On the Recurrence Formulæ of the Lamé Functions of Different Degrees.* By J. L. Sharma, *M.A.**

1. *Introductory.*

ALMOST all the functions occurring in harmonic analysis possess a property of satisfying recurrence formulæ, but curiously enough, so far, no such property has been investigated for Lamé functions, though they belong to the same class of functions. The object of the present paper is to supply this deficiency. It will be seen that the method adopted here for finding recurrence formulæ of the Lamé functions is a generalization of the method given by Professor Whittaker † in his paper "On the Recurrence Formulæ of Mathieu Functions." His method can be very successfully used to find out recurrence formulæ for functions satisfying the partial differential equation, viz.,

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=kV,$$

where k is a constant, but when k is a variable, as is the case with the equation of the ellipsoidal surface harmonics, viz.,

$$\frac{\partial^2 V}{\partial u^2}+\frac{\partial^2 V}{\partial v^2}+n(n+1)[\wp(u)-\wp(v)]V=0,$$

this method can not be used. It will be found that the method given below can be used without any restriction of any kind for all functions occurring in harmonic analysis.

Examples illustrating the application of this method to other functions are omitted simply because they can very easily be obtained by following the procedure given below.

2. *Laplace's Equation.*

Laplace's equation, viz.,

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0, \quad . \quad . \quad . \quad . \quad (2.1)$$

* Communicated by the Author.

† Whittaker, "On the Recurrence Formulæ of Mathieu Functions," Journ. Lond. Math. Soc. iv. part 2, pp. 88–96.

when expressed in terms of uniformized coordinates, becomes

$$[\wp(v)-\wp(w)]\frac{\partial^2 V}{\partial u^2}+[\wp(w)-\wp(u)]\frac{\partial^2 V}{\partial v^2}+[\wp(u)-\wp(v)]\frac{\partial^2 V}{\partial w^2}=0,$$

of which the solutions are

$$\mathrm{E}_n^t(u)\mathrm{E}_n^t(v)\mathrm{E}_n^t(w), \qquad t=1, 2, 3 \ldots (2n+1), \quad n=1, 2 \ldots \infty,$$

where $\mathrm{E}_n^t(u)$ is a Lamé function satisfying the equation

$$\frac{\partial^2 \mathrm{E}}{\partial u^2}=[n(n+1)\wp(u)+\mathrm{B}_t]\mathrm{E}. \quad . \quad . \quad . \quad (2.2)$$

This equation admits $2n+1$ solutions, which are polynomials in $\wp(u)$ and $\wp'(u)$. These solutions, as shown by Halphen *, correspond to $2n+1$ real and distinct values of B satisfying the equation

$$\mathrm{P}.\mathrm{Q}_{e_1}.\mathrm{Q}_{e_2}.\mathrm{Q}_{e_3}=0.$$

For brevity of expression we denote differentiation by dashes, and write

$\wp_r(t)$ for $\sqrt{\wp(t)-e_r}$, $\quad r=1, 2, 3$;
E_n^t for $\mathrm{E}_n^t(u)$.

3. *The Recurrence Formulæ for Functions of Different Degrees.*

We start from the typical fundamental solutions of Laplace's equation in terms of uniformized coordinates, viz.,

$$\mathrm{E}_n^s(u)\mathrm{E}_n^s(v)\mathrm{E}_n^s(w). \quad . \quad . \quad . \quad . \quad . \quad (3.1)$$

From the equation (2.1) it is clear that if V is a solution then $\dfrac{\partial V}{\partial x}$ is a solution, and since

$$\sqrt{(e_3-e_1)(e_3-e_2)}\,\frac{\partial}{\partial x}=\frac{\wp_1(u)\wp_2(u)\wp_3(v)\wp_3(w)}{[\wp(u)-\wp(v)][\wp(u)-\wp(w)]}\frac{\partial}{\partial u}+\frac{\wp_1(v)\wp_2(v)\wp_3(u)\wp_3(w)}{[\wp(v)-\wp(w)][\wp(v)-\wp(u)]}\frac{\partial}{\partial v}+\frac{\wp_1(w)\wp_2(w)\wp_3(u)\wp_3(v)}{[\wp(w)-\wp(u)][\wp(w)-\wp(v)]}\frac{\partial}{\partial w}$$

* Halphen, 'Fonctions elliptiques,' t. ii. (Paris, 1888).

therefore the expression

$$\frac{\wp_1(u)\wp_2(u)\wp_3(v)\wp_3(w)}{[\wp(u)-\wp(v)][\wp(u)-\wp(w)]}\mathrm{E}_n^{s'}(u)\mathrm{E}_n^{s}(v)\mathrm{E}_n^{s}(w)$$
$$+\frac{\wp_1(v)\wp_2(v)\wp_3(u)\wp_3(w)}{[\wp(v)-\wp(w)][\wp(v)-\wp(u)]}\mathrm{E}_n^{s}(u)\mathrm{E}_n^{s'}(v)\mathrm{E}_n^{s}(w)$$
$$+\frac{\wp_1(w)\wp_2(w)\wp_3(u)\wp_3(v)}{[\wp(w)-\wp(u)][\wp(w)-\wp(v)]}\mathrm{E}_n^{s}(u)\mathrm{E}_n^{s}(v)\mathrm{E}_n^{s'}(w)$$
$$\ldots \quad (3.2)$$

is a solution of (2.1), and is therefore expressible as a sum of a series of the fundamental solutions. Therefore the expression (3.2) can be written as

$$\sum_{n=0}^{\infty}\sum_{s=1}^{2n+1} a_{ns}\mathrm{E}_n^{s}(u)\mathrm{E}_n^{s}(v)\mathrm{E}_n^{s}(w). \quad \ldots \quad (3.3)$$

Multiply (3.2) and (3.3) by

$$\mathrm{E}_m^{t}(v)\mathrm{E}_m^{t}(w)[\wp(v)-\wp(w)],$$

and integrate with respect to v and w. On account of the orthogonal property

$$\int_{w_2}^{w_3}\int_{w_1}^{w_2}\mathrm{E}_n^{s}(w)\mathrm{E}_n^{s}(v)\mathrm{E}_m^{t}(w)\mathrm{E}_m^{t}(v)[\wp(v)-\wp(w)]\,dv\,dw=0 \quad \left.\begin{matrix} n\neq m \\ t\neq s \end{matrix}\right\}$$

we get

$$\mathrm{E}_n^{s'}\int_{w_2}^{w_3}\int_{w_1}^{w_2}\frac{\wp_1(u)\wp_2(u)\wp_3(v)\wp_3(w)}{[\wp(u)-\wp(v)][\wp(u)-\wp(w)]}$$
$$\times\mathrm{E}_n^{s}(v)\mathrm{E}_n^{s}(w)\mathrm{E}_m^{t}(w)\mathrm{E}_m^{t}(w)[\wp(v)-\wp(w)]\,dv\,dw$$
$$+\mathrm{E}_n^{s}\int_{w_2}^{w_3}\int_{w_1}^{w_2}\frac{\wp_1(v)\wp_2(v)\wp_3(u)\wp_3(w)}{\{\wp(v)-\wp(u)\}}$$
$$\times\mathrm{E}_n^{s'}(v)\mathrm{E}_n^{s}(w)\mathrm{E}_m^{t}(v)\mathrm{E}_m^{t}(w)\,dv\,dw$$
$$+\mathrm{E}_n^{s}\int_{w_2}^{w_3}\int_{w_1}^{w_2}\frac{\wp_1(w)\wp_2(w)\wp_3(u)\wp_3(v)}{[\wp(u)-\wp(w)]}$$
$$\times\mathrm{E}_n^{s}(v)\mathrm{E}_n^{s'}(w)\mathrm{E}_m^{t}(v)\mathrm{E}_m^{t}(w)\,dv\,dw$$
$$=a_{mt}\mathrm{E}_m^{t}\int_{w_2}^{w_3}\int_{w_1}^{w_2}[\mathrm{E}_m^{t}(v)\mathrm{E}_m^{t}(w)]^2[\wp(v)-\wp(w)]\,dv\,dw.$$

This may be taken as the recurrence formula. Similarly we can deduce a relation between E_m^{t}, $\mathrm{E}_m^{t'}$, E_n^{s}, and we

can thus combine Lamé functions of any two different degrees.

4. *Properties of the Coefficients in the Recurrence Formulæ.*

Writing the recurrence relation in the form

$$\alpha_{ns,\,mt}\mathrm{E}_n^{s'}+\beta_{ns,\,mt}\mathrm{E}_n^{s}=\mu_{ns,\,mt}\mathrm{E}_m^{t}, \quad . \quad . \quad . \quad (4.1)$$

$$\alpha_{mt,\,ns}\mathrm{E}_m^{t'}+\beta_{mt,\,ns}\mathrm{E}_m^{t}=\mu_{mt,\,ns}\mathrm{E}_n^{s}, \quad . \quad . \quad . \quad (4.2)$$

where $\mu_{ns,\,mt}$ and $\mu_{mt,\,ns}$ are constants, we get

$$\alpha_{ns,mt}=\wp_1(u)\wp_2(u)\int_{w_2}^{w_3}\int_{w_1}^{w_2}\frac{\wp_3(v)\wp_3(w)}{[\wp(u)-\wp(v)][\wp(u)-\wp(w)]}\times\mathrm{E}_n^{s}(v)\mathrm{E}_n^{s}(w)\mathrm{E}_m^{t}(v)\mathrm{E}_m^{t}(w)[\wp(v)-\wp(w)]\,dv\,dw, \quad (4.3)$$

$$\beta_{ns,\,mt}=\wp_3(u)\int_{w_2}^{w_3}\int_{w_1}^{w_2}\frac{\wp_1(v)\wp_2(v)\wp_3(w)}{\wp(v)-\wp(u)}\times\mathrm{E}_n^{s'}(v)\mathrm{E}_n^{s}(w)\mathrm{E}_m^{t}(v)\mathrm{E}_m^{t}(w)\,dv\,dw$$

$$+\wp_3(u)\int_{w_2}^{w_3}\int_{w_1}^{w_2}\frac{\wp_1(w)\wp_2(w)\wp_3(v)}{\wp(u)-\wp(w)}\times\mathrm{E}_n^{s}(v)\mathrm{E}_n^{s'}(w)\mathrm{E}_m^{t}(v)\mathrm{E}_m^{t}(w)\,dv\,dw, \quad (4.4)$$

$$\alpha_{mt,\,ns}=\wp_1(u)\wp_2(u)\int_{w_2}^{w_3}\int_{w_1}^{w_2}\frac{\wp_3(v)\wp_3(w)[\wp(v)-\wp(w)]}{[\wp(u)-\wp(v)][\wp(u)-\wp(w)]}\times\mathrm{E}_n^{s}(v)\mathrm{E}_n^{s}(w)\mathrm{E}_m^{t}(v)\mathrm{E}_m^{t}(w)\,dv\,dw, \quad (4.5)$$

$$\beta_{mt,\,ns}=\wp_3(u)\int_{w_2}^{w_3}\int_{w_1}^{w_2}\frac{\wp_1(v)\wp_2(v)\wp_3(w)}{\wp(v)-\wp(u)}\times\mathrm{E}_n^{s}(v)\mathrm{E}_n^{s}(w)\mathrm{E}_m^{t'}(v)\mathrm{E}_m^{t}(w)\,dv\,dw$$

$$+\wp_3(u)\int_{w_2}^{w_3}\int_{w_1}^{w_2}\frac{\wp_1(w)\wp_2(w)\wp_3(v)}{\wp(u)-\wp(w)}\times\mathrm{E}_n^{s}(v)\mathrm{E}_n^{s}(w)\mathrm{E}_m^{t}(v)\mathrm{E}_m^{t'}(w)\,dv\,dw. \quad (4.6)$$

From these it is clear that

$$\alpha_{ns,\,mt}=\alpha_{mt,\,ns}. \quad . \quad . \quad . \quad . \quad . \quad (4.7)$$

Again, since

$$\frac{d}{dv}\{\mathrm{E}_n^{s}(v)\mathrm{E}_m^{t}(v)\}=\mathrm{E}_n^{s'}(v)\mathrm{E}_m^{t}(v)+\mathrm{E}_n^{s}(v)\mathrm{E}_m^{t'}(v),$$

therefore, by adding (4.4) and (4.6) and integrating by parts one term with respect to v and the other with respect

to w, and differentiating (4.3) with respect to u, we obtain the equation

$$\beta_{ns,\,mt}+\beta_{mt,\,ns}+\frac{d\alpha_{ns,\,mt}}{du}=0. \quad . \quad . \quad . \quad (4.8)$$

Differentiating (4.2) we get

$$\alpha'_{mt,\,ns}\mathrm{E}^{t\prime}_{m}+\alpha_{mt,\,ns}\mathrm{E}^{t\prime\prime}_{m}+\beta'_{mt,\,ns}\mathrm{E}^{t}_{m}+\beta_{mt,\,ns}\mathrm{E}^{t\prime}_{m}=\mu_{mt,\,ns}\mathrm{E}^{s\prime}_{n}.$$

Now eliminating E^{s}_{n} and $\mathrm{E}^{s\prime}_{n}$ from this, (4.1) and (4.2), and using (4.7) and (4.8), we get

$$(\alpha_{mt,\,ns})^2\mathrm{E}^{t\prime\prime}_{m}+(\beta'_{mt,\,ns}\alpha_{mt,\,ns}-\mu_{mt,\,ns}\mu_{ns,\,mt}+\beta_{mt,\,ns}\beta_{ns,\,mt})\mathrm{E}^{t}_{m}=0,$$

and comparing it with the equation (2.2) we find

$$(\alpha_{mt,\,ns})^2[m(m+1)\wp(u)+\mathrm{B}_t]+(\beta'_{mt,\,ns}\alpha_{mt,\,ns}-\mu_{mt,\,ns}\mu_{ns,\,mt}+\beta_{mt,\,ns}\beta_{ns,\,mt})=0. \quad (4.9)$$

Again, write

$$\mathrm{U}=\alpha_{ns,\,mt}\mathrm{E}^{s\prime}_{n}+\beta_{ns,\,mt}\mathrm{E}^{s}_{n},$$

so that U is a constant multiple of E^{t}_{m}.

Differentiating it twice and replacing $\mathrm{E}^{t\prime\prime}_{m}$ by $[m(m+1)\wp(u)+\mathrm{B}_t]\mathrm{E}^{t}_{m}$ at every stage, we get

$$\begin{aligned}\mathrm{U}''=&\mathrm{E}^{s\prime}_{n}\{\alpha_{ns,\,mt}(n(n+1)\wp(u)+\mathrm{B}_s)+\alpha''_{ns,\,mt}+2\beta'_{ns,\,mt}\}\\&+\mathrm{E}^{s}_{n}[(2\alpha'_{ns,\,mt}+\beta_{ns,\,mt})\{n(n+1)\wp(u)+\mathrm{B}_s\}\\&+\alpha_{ns,\,mt}.n(n+1)\wp'(u)+\beta''_{ns,\,mt}]\\=&[\alpha_{ns,\,mt}\mathrm{E}^{s\prime}_{n}+\beta_{ns,\,mt}\mathrm{E}^{s}_{n}][m(m+1)\wp(u)+\mathrm{B}_t],\end{aligned}$$

since U satisfies the same equation as E^{t}_{m}.

Hence finally

$$\begin{aligned}\mathrm{E}^{s\prime}_{n}[&\alpha_{ns,\,mt}\{n(n+1)\wp(u)+\mathrm{B}_s-m(m+1)\wp(u)-\mathrm{B}_t\}\\&+\alpha''_{ns,\,mt}+2\beta'_{ns,\,mt}]+\mathrm{E}^{s}_{n}[2\alpha'_{ns,\,mt}\{n(n+1)\wp(u)+\mathrm{B}_s\}\\&+\beta_{ns,\,mt}\{n(n+1)\wp(u)+\mathrm{B}_s-m(m+1)\wp(u)-\mathrm{B}_t\}\\&+2\beta''_{ns,\,mt}+\alpha_{ns,\,mt}.n(n+1)\wp'(u)]=0.\end{aligned}$$

This may be written as

$$\mathrm{PE}^{s\prime}_{n}+\mathrm{QE}^{s}_{n}=0, \quad . \quad . \quad . \quad . \quad . \quad (4.91)$$

where

$$\begin{aligned}\mathrm{P}=&\alpha_{ns,\,mt}[n(n+1)\wp(u)+\mathrm{B}_s-m(m+1)\wp(u)-\mathrm{B}_t]\\&+\alpha''_{ns,\,mt}+2\beta'_{ns,\,mt}\end{aligned}$$

and

$$Q=[2\alpha'_{ns,\,mt}\{n(n+1)\wp(u)+B_s\}+\beta_{ns,\,mt}$$
$$\times\{n(n+1)\wp(u)+B_s-m(m+1)\wp(u)-B_t\}$$
$$+2\beta''_{ns,\,mt}+\alpha_{ns,\,mt}.\,n(n+1)\wp'(u)].$$

From (4.8)

$$\beta_{mt,\,ns}=-\alpha'_{ns,\,mt}-\beta_{ns,\,mt},$$

and

$$\beta'_{mt,\,ns}=-\alpha''_{ns,\,mt}-\beta'_{ns,\,mt}.$$

Substitute these in (4.9), we get

$$\alpha''_{ns,\,mt}\alpha_{ns,\,mt}+\alpha_{ns,\,m}\beta'_{ns,\,mt}+\alpha'_{ns,\,mt}\beta_{ns,\,mt}+\mu_{ns,\,mt}.\mu_{mt,\,ns}$$
$$+\beta^2{}_{ns,\,mt}-\alpha^2{}_{ns,\,mt}\{m(m+1)\wp(u)+B_t\}=0,$$

of which the differential coefficient may be written as

$$\frac{dP}{du}+\frac{\alpha'_{ns,\,mt}+\beta_{ns,\,mt}}{\alpha_{ns,\,mt}}P-Q=0. \quad . \quad . \quad (4.92)$$

The equations (4.91) and (4·92) cannot hold at the same time unless

$$P=0 \text{ and } Q=0,$$

i. e., $\alpha_{ns,\,mt}\{n(n+1)\wp(u)+B_s-m(m+1)\wp(u)-B_t\}$
$$+\alpha''_{ns,\,mt}+2\beta'_{ns,\,mt}=0, \quad (4.93)$$

and $2\alpha'_{ns,\,mt}[n(n+1)\wp(u)+B_s]+\beta_{ns,\,mt}$
$$\times\{n(n+1)\wp(u)+B_s-m(m+1)\wp(u)-B_t\}$$
$$+2\beta''_{ns,\,mt}+\alpha_{ns,\,mt}.\,n(n+1)\wp'(u)=0. \quad (4.94)$$

There are thus five equations, namely (4.7), (4.8), (4.9), (4.93), and (4.94), to determine the four quantities, but it can be seen that they are not all independent.

5. *The Differential Equation satisfied by the Coefficients.*

Eliminating $\beta_{ns,\,mt}$ from (4.93) and (4.94) we get a differential equation for $\alpha_{,s,\,mt}$, namely (writing α for $\alpha_{ns,\,mt}$)

$$(p-q)\alpha^{iv}-(p'-q')\alpha'''-2(p^2-q^2)\alpha''$$
$$+\{-pp'+5p'q-5q'p+qq'\}\alpha'$$
$$+[(p-q)^3+(p'^2-q'^2)-(p-q)(p''+q'')]\alpha=0,$$

where p stands for $n(n+1)\wp(u)+B_s$ and q stands for $m(m+1)\wp(u)+B_t$. But a close examination will show

that this is the same equation which is satisfied by the product of the solutions of

$$y''=[n(n+1)\wp(u)+B_s]y$$

and

$$y''=[m(m+1)\wp(u)+B_t]y.$$

Hence it follows that $\alpha_{ns,\,mt}$ is of the form

$$A_1E_n^sE_m^t+B_1E_n^sF_m^t+C_1E_m^tF_n^s+D_1F_m^tF_n^s,$$

where F_m^t is the second solution of the Lamé equation.

From (4.93) we get

$$\beta_{ns,\,mt}=-\{A_1E_n^{s\prime}E_m^t+B_1E_n^{s\prime}F_m^t+C_1F_n^{s\prime}E_m^t+D_1F_n^{s\prime}F_m^t\}.$$

Substitute these values in (4.1). Since E_m^t is periodic and

$$E_n^{s\prime}F_n^s-E_n^sF_n^{s\prime}=2n+1,$$

we find that

$$D_1=0$$

and

$$\mu_{ns,\,mt}=(2n+1)C_1.$$

Hence, finally, we get

$$\alpha_{ns,\,mt}=A_1E_n^sE_m^t+B_1E_n^sF_m^t+C_1E_m^tF_n^s,$$

$$\beta_{ns,\,mt}=-(A_1E_n^{s\prime}E_m^t+B_1E_n^{s\prime}F_m^t+C_1E_m^tF_n^{s\prime}),$$

$$\mu_{ns,\,mt}=(2n+1)C_1.$$

Similarly we can deduce the coefficients of (4.2) from the equations (4.7) and (4.8), and they are

$$\alpha_{mt,\,ns}=A_1E_m^tE_n^s+B_1E_n^sF_m^t+C_1E_m^tF_n^s,$$

$$\beta_{mt,\,ns}=-(A_1E_m^{t\prime}E_n^s+B_1E_n^sF_m^{t\prime}+C_1E_m^{t\prime}F_n^s),$$

$$\mu_{mt,\,ns}=(2m+1)B_1.$$

It can further be seen that if $m=n$ we get the recurrence formulæ of the Lamé functions of the same degree.

XCVI. *X-ray Levels and Atomic Constants.*—Part II. *By* H. R. Robinson, *F.R.S., Queen Mary College (University of London)* †.

§ 1. THERE are now available many precise measurements of the energies of X-ray electrons, obtained with the magnetic spectrometer. References

† Communicated by the Author.

and data are to be found in two recent papers †, which contain also summaries of results selected from three earlier communications ‡.

These measurements may now profitably be discussed in the light of our latest knowledge of the probable values of the atomic constants used in interpreting them. As is well known, the general problem of the fundamental constants has been and is being very thoroughly discussed, particularly by Prof. R. T. Birge, and the present discussion will therefore be limited to the most direct application of the electron energy measurements.

In the earlier applications of the magnetic spectrometer the observed kinetic energies of the electrons (deduced from magnetic deflexions) were compared with the energies calculated from the Einstein photoelectric equation—*i. e.*, with $h(\nu_0-\nu_A)$; here ν_0 is the frequency of the primary X-rays and ν_A the critical absorption frequency for the level from which the electron is extracted. The values of ν_0 and ν_A were taken from standard X-ray spectroscopic work—generally from the tables published by Siegbahn—and the former at least were known with very high accuracy in terms of the conventional X-unit. Many of the critical absorption frequencies were much less accurately known—indeed, some even of the earlier work on X-ray electrons was sufficiently accurate to reveal inconsistencies in accepted X-ray data. Much of the photoelectric work has, for technical reasons, been done with moderately soft primary X-rays; hence many of the levels explored have not been among those which are most accurately measurable in X-ray spectroscopy. During the past few years much accurate experimental work has been done upon absorption edges and their structure, and Siegbahn and his school in particular have done much to systematize our knowledge of their finer details §. There is, however, no very direct way in which we can derive from X-ray measurements the

† Robinson, Phil. Mag. ser. 7, xviii. p. 1086 (1934); Robinson and Clews, Proc. Roy. Soc. A, cxlix. p. 587 (1935)—later quoted as R. 1 and R. 2 respectively.

‡ Kretschmar, Phys. Rev. xliii. p. 417 (1933); Robinson, Andrews, and Irons, Proc. Roy. Soc. A, cxliii. p. 48 (1933); Robinson, Proc. Phys. Soc. xlvi. p. 693 (1934).

§ There is much relevant discussion of this in a recent paper by Sandström (Phil. Mag. xxii. p. 171 (1936)), which was received when the greater part of the present paper had been written.

appropriate value of ν_A for insertion in our photoelectric equation. This introduces a serious uncertainty into the direct application of the photoelectric results to the evaluation of atomic constants.

In the later work (see especially R. 1, p. 1097) a method has been adopted which eliminates this uncertainty by rendering unnecessary any appeal to spectroscopically measured absorption limits. The method is simply that of measuring the energies of the electrons expelled from the same levels by a number of primary X radiations covering a suitable range of frequency, and comparing the energy differences with the corresponding frequency differences. It will be obvious that this method also eliminates a minor source of inaccuracy which was present in the earlier work—the effect on the measured energies of the work required to draw the photoelectrons from the surface of the target. No correction was applied for this in the past, as it was assumed that the effect was small, not only in relation to the measured energies but also in comparison with other sources of error. Finally, it may be recalled that in the new work all the experiments have been made with the same apparatus throughout; the difference method therefore, though it does not eliminate the effect of a constant zero error, does at least reduce it.

The new method of handling the photoelectric measurements makes only a small difference in the final result when the different series are averaged—a fact which says much for the general all-round accuracy of modern X-ray data. With the new method, however, there is, as would be expected, improved internal consistency within the series of measurements, and the final result should rank more highly as a contribution to the problem of the atomic constants.

§ 2. The quantities measured in the magnetic spectrometry of photoelectrons are r and H, r being the radius of curvature of the electron path in a field H gauss perpendicular to the plane of the path. The kinetic energy of the electron is deduced from the product (rH), using the formula

$$E_{kin}=m_0c^2[\{1+(r\mathrm{H}.e/m_0c)^2\}^{\frac{1}{2}}-1],$$

which for slow electrons reduces to the non-relativistic form

$$E_{kin}=\tfrac{1}{2}(rH)^2\times e/m_0\times e.$$

The difference between two of these E values has to be equated to $h(\nu_1-\nu_2)$, where ν_1 and ν_2 are two primary X-ray frequencies, and we should find

$$h(\nu_1-\nu_2)\doteq\tfrac{1}{2}e/m_0\times e\{(r_1H_1)^2-(r_2H_2)^2\}\dagger$$

—a relation between measured quantities and the product $(e/m_0\times e/h)$. So far it has been assumed that ν_1 and ν_2 are known absolutely in sec.$^{-1}$. The X-ray frequencies are not, however, measured directly; they are deduced from the measured wave-lengths, and are usually expressed in terms of the Rydberg frequency as unit (ν/R, or wave number divided by Rydberg number, 109737). To get the true values of ν/R we need to divide the tabulated values by ρ, the ratio of the true to the conventional value of the grating space of calcite: ρ, in other words, is the value of the X unit expressed in 10^{-11} cm.

Our photoelectric experiments therefore furnish a method for measuring $e/m_0\times e/h\times\rho$—or, from the way in which ρ can be expressed in terms of accurately known constants, for measuring $e/m_0.e^{4/3}/h$. The procedure in papers R1 and R2 was to calculate E from the magnetic deflexion and to transform it into an equivalent frequency by Planck's relation, assuming accepted values of e/m_0 and e/h. Values of this equivalent frequency, calculated in terms of the Rydberg frequency Rc and denoted by ν^*/R, were tabulated for photoelectrons ejected from a number of levels by three different homogeneous X radiations in turn. The X radiations used were the Kα_1 rays of chromium (24), copper (29), and molybdenum (42).

The values of ν^*/R were based on the assumptions $c=2{\cdot}998\times10^{10}$ cm.sec.$^{-1}$, $e/m_0=1{\cdot}757\times10^{7}$ e.m.u.gm.$^{-1}$, and $e/h=7{\cdot}2827\times10^{16}$ e.s.u. erg^{-1} sec.$^{-1}$. The values taken for the primary X-ray frequencies were

$$\begin{aligned}\text{Cr K}\alpha_1\ \nu/\text{R}&=\ 398{\cdot}80,\\ \text{Cu K}\alpha_1\ \nu/\text{R}&=\ 592{\cdot}74,\\ \text{MoK}\alpha_1\ \nu/\text{R}&=1287{\cdot}42.\end{aligned}$$

† The equation is so written here for simplicity: in the actual calculations the full relativistic formula for E is of course used.

The corresponding wave-lengths in X units are 2285·03 (Eriksson, *Z. Phys.* xlviii. p. 360 (1928)) ; $1537{\cdot}39_5$ (Siegbahn, *Ark. Mat., Astr. o. Fys.* (A), xxi. no. 21 (1929), and Wennerlöf, *ibid.* xxii. no. 8 (1930)) ; and 707·83 (Larsson, Phil. Mag. (7) iii. p. 1136 (1927) ; see also data in Compton and Allison, ' X-Rays in Theory and Experiment,' p. 689 (Macmillan, 1935)). The (ν/R)s are conventional values, the X-unit being assumed to be 10^{-11} cm. exactly—*i. e.*, the correcting factor $1/\rho$ has not been applied. They are, however, known very accurately in terms of the conventional X-ray units. As a result of the use of a difference method the accuracy of our deductions from the X-ray data would be most seriously affected by an error in the shortest of the three wave-lengths, but fortunately, as Compton and Allison (*loc. cit.*) point out, the Bragg angle for first-order reflexion of MoKα_1 from calcite is now one of the most accurately known of all physical constants. Further, as a result of recent experiments bearing both directly and indirectly on the value of ρ our knowledge of the *absolute* values of X-ray frequencies is now very complete and accurate.

The direct determination of ρ depends in effect upon the measurement of the ratio $\dfrac{\lambda_g}{2d_1 \sin \theta_1}$, where λ_g is the " grating " wave-length of an X-radiation, d_1 the assumed effective grating space of calcite for first-order reflexion, and θ_1 the corresponding Bragg angle. d_1 is less than d_x, the " calculated true " grating space, by about 0·4 X-unit—the correction for refraction.

The most recent and reliable determinations of $\dfrac{\lambda_g}{2d_1 \sin \theta_1}$ are those of Bäcklin †, Södermann ‡, and Bearden §. The results as given are :—

Bäcklin.....	1·00218
Södermann	1·00225
Bearden	1·00248

† Bäcklin, ' Nature,' cxxxv. p. 32 (1935) ; *Zeits. Phys.* xciii. p. 450 (1935).

‡ Södermann, ' Nature,' cxxxv. p. 67 (1935) ; Nov. Act. Reg. Soc. Sci. Upsal. (ser. 4) ix. no. 8 (1935).

§ Bearden, Phys. Rev. xlviii. p. 385 (1935).

In comparing them it must be remembered that the figures are based on different assumed values of d. Bäcklin and Södermann take $d_\infty = 3{\cdot}02904$ Å. (Compton, Beets, and DeFoe's † value), while Bearden ‡ uses his own value of 3·02810 (both at 18° C.). The corresponding Moseley-Siegbahn § value is 3·02945 (with $d_1 = 3{\cdot}02904$), so none of the above ratios corresponds exactly to the ρ defined earlier in this paper. It is very satisfactory to note that (1) the difference between the Compton and Bearden values of d is almost completely accounted for by the different values of e and c used in the computations, and (2) the results quoted above by Bäcklin and Bearden agree to within 1 part in 100,000 when they are reduced to the same value of d. Södermann's value of ρ, which is roughly 1 part in 15,000 greater than the others, is, like Bäcklin's, included in Bearden's final figure, which is really the weighted mean of a large number of results.

We may therefore take ρ as 1·0020 (=ratio of ruled grating to *Siegbahn* wave-lengths), probably correct to within 1 part in 30,000, and $e = 4{\cdot}803_5 \times 10^{-10}$ abs.e.s.u., probably correct to about 1 part in 10,000.

An earlier objection to this relatively high value of ρ—the difficulty of reconciling it with the "oil-drop" value of e—has been deprived of its force by the recent work of Kellström ‖, Bond ¶, and Bäcklin and Flemberg **. Kellström and Bond have independently redetermined the coefficient of viscosity of air, their values at 23° C. being in very close agreement with one another ($1834{\cdot}8 \pm 3{\cdot}0$ and $1834{\cdot}7 \pm 0{\cdot}8 \times 10^{-7}$ respectively), and just over 0·65 per cent. greater than the value used by Millikan in his oil-drop work. The new value of the viscosity, combined with Millikan's oil-drop measurements, gives $e = 4{\cdot}816 \times 10^{-10}$ e.s.u., which is appreciably larger than the mean X-ray value. On the other hand, Bäcklin and Flemberg have repeated the oil-drop measurements, using a wide range of gas pressures in order to test the departures from Stokes's law, and their data yield

† Compton, Beets, and De Foe, Phys. Rev. xxv. p. 625 (1925).
‡ Bearden, Phys. Rev. xxxviii. p. 2089 (1931)
§ Siegbahn, 'Spektroskopie der Röntgentrahlen': Springer, Berlin (1931).
‖ Kellström, 'Nature,' cxxxvi. p. 682 (1935).
¶ Bond, 'Nature,' cxxxvii. p. 1031 (1936).
** Bäcklin and Flemberg, 'Nature,' cxxxvii. p. 655 (1936).

$e=4{\cdot}800\times10^{-10}$—a value only slighter lower than the X-ray value—when they are used with the Kellström-Bond value of the viscosity. Bäcklin and Flemberg also quote results from a recent paper by Friesen † (which I have not been able to consult) in support of the new work.

§ 3. In the papers R1 and R2 the results are divided into three sets, corresponding to the three possible ways of combining the results of the experiments with three different primary X-radiations. Taking the mean value for each set, the ν^*/R (photoelectron equivalent frequencies in Rydberg units, calculated with the constants quoted in § 2 above) exceed the ν/R (uncorrected X-ray) values by $0{\cdot}1_2$, $0{\cdot}2_4$, and $0{\cdot}2_4$ per cent. respectively. Most of the plates have recently been remeasured—the results showing no appreciable differences from the first measurements—and the figures have been more systematically averaged. It is found that the average excess of the ν^*/R over the corresponding ν/R values is 0·17 per cent. It is not easy to assess the probable error, but I think the true value of this excess lies between 0·10 and 0·25 per cent.

Taking the mean value of 0·17 per cent., the direct numerical expression of the photoelectric measurements is that the value of $e/m_0.e/h.\rho$ is $0{\cdot}9983\times1{\cdot}757\times10^7\times7{\cdot}2827\times10^{16}=1{\cdot}277_4\times10^{24}$—in the usual mixed units, e/m_0 in e.m.u.gm.$^{-1}$ and e in e.s.u.

The photoelectric measurements alone will take us no further than this, but in view of the accuracy with which ρ and e/m_0 may now be presumed known, it is worth while treating this result as an indirect determination of e/h ‡.

Putting $\rho=1{\cdot}0020$, and, following Birge, $e/m_0=1{\cdot}7576\times10^7$, we get $e/h=7{\cdot}253_3\times10^{16}$—or h/e, which is more frequently quoted, $=1{\cdot}378_7\times10^{-17}$.

These agree well (within the experimental error, in fact) with the values deduced from the equation

$$\frac{e}{h}=\sqrt{\frac{\mathrm{R}c^2}{2\pi^2e^2}\cdot\frac{e}{m_0}}.$$

If we put $\mathrm{R}=109737{\cdot}4$, $c=2{\cdot}9978\times10^{10}$, $e/m_0=1{\cdot}7576\times10^7$, and $e=4{\cdot}8035\times10^{-10}$, we get $e/h=7{\cdot}2468\times10^{16}$,

† Friesen, *Ups. Univ. Årskr.* p. 14 (1935).
‡ *Cf.* Birge, 'Nature,' cxxxvii. p. 187 (1936).

which is less than 1 part in 1000 smaller than our "photoelectric" value.

From the recent work of Kirkpatrick and Ross † on the short-wave limit of the continuous X-ray spectrum, the minimum excitation potential difference for $\lambda=558{\cdot}28$ X-units is 22088 international volts. Taking $c=2{\cdot}9978\times10^{10}$ cm. sec.$^{-1}$, $\rho=1{\cdot}0020$, and 1 international volt$=1{\cdot}0004$ absolute volts, this gives $e/h=7{\cdot}2705\times10^{16}$.

Again, in the more recent paper of Schaitberger ‡ we find V$=9644{\cdot}3$ international volts for $\lambda=1279{\cdot}17$ X-units. This gives $e/h=7{\cdot}2688\times10^{16}$.

We may tabulate thus :—

Method.	e/h.	h/e.
Magnetic spectrometer	$7{\cdot}253\times10^{16}$	$1{\cdot}379\times10^{-17}$
From Rydberg constant	$7{\cdot}2468\times10^{16}$	$1{\cdot}3799\times10^{-17}$
Inverse photo-effect (Kirkpatrick and Ross).	$7{\cdot}2705\times10^{16}$	$1{\cdot}3754\times10^{-17}$
Inverse photo-effect (Schaitberger)..	$7{\cdot}2688\times10^{16}$	$1{\cdot}3757\times10^{-17}$

Summary.

The latest results obtained with the magnetic spectrometer indicate that $e/m_0\times e/h\times\rho=1{\cdot}277_4\times10^{24}$, where ρ is the ratio of the true to the conventional Moseley-Siegbahn wave-length of an X-radiation. If recent and apparently reliable values are assigned to e/m_0 and ρ, this result leads to a value of e/h which agrees well with that deduced from the Rydberg constant and less well with the values obtained in recent experiments on the inverse photoelectric effect.

Acknowledgment.

It is a pleasure to thank Professors R. T. Birge and A. E. Ruark for their kindness in sending me some of their results, and for friendly and helpful discussion of the details of mine.

Queen Mary College (University of London),
July 1936.

† Kirkpatrick and Ross, Phys. Rev. xlv. p. 454 (1934).
‡ Schaitberger, *Ann. der Physik*, xxiv. p. 84 (1935).

XCVII. *An Appraisal of Empirical Formulæ relating Equilibrium Vibration Frequency and Distance in Non-Hydride, Di-Atom Nuclei.* *By* C. H. Douglas Clark, *D.Sc., A.R.C.S., A.I.C., D.I.C., and* John L. Stoves, *M.Sc., University of Leeds.*

I. *Introduction.*

IN a letter to the Editor of 'Nature' [1], the authors have communicated some preliminary results of an attempt to ascertain, amongst various suggested empirical relations between nuclear vibration frequency and internuclear distance of non-hydride di-atoms, which reproduces the experimental findings most accurately. It was reported that the formula suggested by Douglas Clark [2] was the best in the KK period, for both ground and excited states of all di-atoms for which the necessary data were available. The purpose of the present memoir is to show the results of calculation in the KK period in greater detail, and to extend the enquiry to other periods.

Relations of the type here considered assume significance in view of the preliminary theoretical attack on the problem by Newing [3] from the standpoint of wave mechanics. A formula has been derived, whose form is consistent with the author's conclusion that "we shall expect a law of the type that has been suggested by Badger and by Clark." Further analysis has been promised. In the meantime, therefore, it seems desirable to examine the applicability of the suggested formulæ as carefully as possible.

The relations are fundamentally important in connexion with the construction of potential energy-nuclear distance curves.

II. *Suggested Modifications of Morse's Rule.*

Following earlier proposals of Birge and Mecke in 1925, Morse [4] in 1929 made the significant suggestion that the relation between r_e, the internuclear distance of a di-atom in Ångström units, and ω_e, the equilibrium nuclear vibration frequency in waves per cm., might take the form

$$\omega_e r_e^3 = 3000. \quad . \quad . \quad . \quad . \quad . \quad . \quad (M)$$

* Communicated by Dr. C. H. Douglas Clark.

This may be called the M formula. Morse showed that it could be applied to a certain number of di-atoms with considerable precision. Jevons [5] in 1932 stated that "Morse's rule is the only known means of estimating r_e for a given electronic state of a molecule, when ω_e has been determined for that state, but no value of r_e is known for any electronic state of the molecule." It was, however, fairly widely known at that time that the equation might be subject to considerable error, and here it should be observed that Morse recognized a limitation, in the sense that the masses of the two atoms concerned should not deviate too greatly, and that he introduced an asymmetry correction to allow for such departures. No one has ever, apparently, adopted the suggestion, or used such a correction. Indeed, the idea was largely discounted by the observation that the M formula gave large errors of the order of 25 per cent. in estimating r_e from ω_e in the case of the quite symmetrical di-atom LiLi [2]. It was further established that Morse's function, far from being constant, was really a periodic group function, and that this was true both for hydride and non-hydride di-atoms [6]. The periodicity of Morse's function suggested that an appropriate correction might take the form of an allowance made for the position of a given di-atom with respect to its molecular period and group. The first systematic attempt to analyse the available data by Douglas Clark [2] in 1934 showed that the M formula was not generally satisfactory, and that the errors were sometimes very large, as in the case of LiLi noted above. It was found that the formula only came near the truth in those molecular groups where the periodic error crossed the zero line, near Groups IX. and X. Since Morse had mainly considered symmetrical molecules in this region, it is not difficult to understand why the misconception arose that the correction must take the form of an asymmetry factor.

The classification of di-atoms into groups and periods was suggested by Clark [7] in 1932, and extended somewhat in 1933 [8]. A much fuller paper appeared last year [9]. The group number of a molecule is denoted by n, and is equal to the sum of the numbers of "valency" electrons of the constituent atoms. The period depends on the non-bonding system, and may be denoted by symbols such as KK, KL, LL . . . It seemed natural to

attempt to utilize this classification in finding a modification of the M formula. It is a striking fact that all attempts to modify the rule have depended upon the periodic classification in one way or another.

Badger [10] adopted similar views in 1934, and reached the conclusion that " the class or group to which a molecule belongs depends only on the rows in the periodic system in which the elements composing it appear." The rows, of course, define the chemical periods, but no notice was taken by Badger of valency groups in this statement. Proceeding on these lines, he derived the empirical relation

$$k_0(r_e - d_{ij}) = 1{\cdot}86 \times 10^5, \quad . \quad . \quad . \quad . \quad . \quad (B)$$

where k_0 is the " bond constant," given by a pendulum-type formula $\omega_e c = (1/2\pi)/(k_0/\mu)^{\frac{1}{2}}$, c is the velocity of light in cm. per sec., d_{ij} is a constant " characteristic of all diatomic molecules made up of one element in the ith row and one in the jth row of the periodic system," and μ is the " reduced mass," equal to $1{\cdot}649 \times 10^{-24} A_1 A_2/(A_1 + A_2)$, where A_1, A_2 are the atomic weights concerned on the O=16 scale. This formula reduces to

$$r_e = d_{ij} + 1{\cdot}737 \times 10^{-6} \left(\frac{1}{\omega_e^2 \mu} \right)^{\frac{1}{3}}, \quad . \quad . \quad . \quad (B\,1)$$

which may be termed the B formula.

In 1935, Badger [11] attempted to improve the equation (B) by re-writing it in the form

$$k_0(r_e - d'_{ij})^3 = C_{ij}, \quad . \quad . \quad . \quad . \quad . \quad (B')$$

where C_{ij} varied in different periods. This reduces to

$$r_e = d'_{ij} + 3{\cdot}043 \times 10^{-8} \left(\frac{C_{ij}}{\omega_e^2 \mu} \right)^{\frac{1}{3}}, \quad . \quad . \quad (B'\,1)$$

which may be called the B′ formula.

In an independent communication, Douglas Clark [2] modified the M formula in 1934 on the basis of his periodic group classification for di-atoms, by introducing the Group number n (defined above), k a constant for the period concerned, and k' a small correction for molecule-ions, as follows :—

$$\omega_e r_e^3 n^{\frac{1}{2}} = k - k'. \quad . \quad . \quad . \quad . \quad . \quad . \quad (C)$$

The C formula is the only one which has so far made an allowance for the case of ionized molecules. This paper

shows it to be applicable to the KK period with an average error of about 1·2 per cent., but certain deviations from experiment occur in later periods, and it is suggested that k is a structurally-sensitive number, assuming different values k_{qr}, according to the nature of the valency groups of the atoms concerned. An atom having only s electrons in its outermost level is here termed an s-atom, whilst if both s and p, or both s and d, electrons are present it is called a p-atom or a d-atom respectively for brevity in representation (see Table XII.). Four cases are here considered :

Combination.			Designation of period constant.	
s-	atom with s-	atom ..... k_{ss}	for ss	di-atom.
s-	atom with p-	atom ..... k_{sp}	for sp	di-atom.
p-	atom with p-	atom ..... k_{pp}	for pp	di-atom.
p-	atom with d-	atom ..... k_{pd}	for pd	di-atom.

The C formula gives

$$r_e = \left(\frac{k-k'}{\omega_e n^{\frac{1}{2}}}\right)^{\frac{1}{3}}, \quad \ldots\ldots \quad (C1)$$

which with the suggested modification becomes

$$r_e = \left(\frac{k_{qr}-k'_{qr}}{\omega_e n^{\frac{1}{2}}}\right)^{\frac{1}{3}}, \quad \ldots\ldots \quad (C')$$

here styled the C′ formula.

A further modification of Morse's rule was made by Allen and Longair [12] in 1935. The form of the equation is that of the C formula, with n replaced by μ, the reduced mass of the di-atom,

$$\omega_e r_e^3 \mu^{\frac{1}{2}} = K, \quad \ldots\ldots \quad (A)$$

where K is a Period constant. It was pointed out that for isotopic molecules, the r_e's might be closely alike, whilst the ω_e's were different, so that the C formula would not differentiate between them, since their n's were the same. In the data considered, however, the only two non-hydride isotopic di-atoms involved are $B^{10}O$ and $B^{11}O$, and it is pointed out elsewhere [1] that the errors in estimating r_e in both cases are within the average experimental error in the period (KK) (see Table I.). Hence, so far as the available evidence goes, the objection loses much of its weight, at any rate for non-hydride di-atoms :

it might be more serious for hydrides, and especially for HH, HD and DD, but since we are only concerned here with non-hydrides, this point is left over for further consideration. The A formula gives

$$r_e = \left(\frac{K}{\omega_e \mu^{\frac{1}{2}}}\right)^{\frac{1}{3}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (A1)$$

Summarizing, we may say that the constants involved in the five formulæ considered are as in the following scheme (see Table IX. for numerical values) :—

Formula.	Constants.		
	Molecular.	Group.	Period.
A.....	μ	..	K
B.....	μ	..	d_{ij}
B′	μ	..	d'_{ij}, C_{ij}
C	..	n	k (k')*
C′	..	n	k_{qr} (k'_{qr})*

* For ionized molecules only.

In the following Tables, where the percentage errors of the five formulæ, A, B, B′, C, C′, are compared, the values of these constants, referred to r_e in Å.U. and μ in 10^{-24} C.G.S. units, are given with each table. Data are mainly collected from Appendix II. of [5], and from 'Molekülspektren' (I. Tabellen) by H. Sponer (Berlin, Julius Springer, 1935). This book contains an extensive bibliography on pp. 57–62, 65–70, and 145.

III. *Results in the* KK *Period.*

Table I. shows the results of calculation in the KK period (see fig. 1). Large errors, greater than 10 per cent., creep in with the A formula for LiLi. As noted by Allen and Longair, "lithium for this formula seems to fall into a different class of molecules." This may, possibly, be associated with the fact that it is the only *ss*-di-atom in the table, and may require a different constant, according to the method suggested by the present writers. The B and C formulæ fit better throughout the whole period than the A formula, the mean percentage errors

TABLE I.—Calculated Results in the KK Period.

K=10,602 ; $d_{ij}=d'_{ij}=0{\cdot}680$; $C_{ij}=0{\cdot}186$; $k=9{,}700$; $k_{ss}=9{,}940$; $k_{sp}=9{,}710$; $k_{pp}=9{,}680$; $k'=k'_{ss}=k'_{sp}=k'_{pp}=0$.

Di-atom.	State.	ω_e.	μ.	n.	r_e (expt.).	r_e (calc.). A.	B.	C.	B′.	C′.	Percentage errors. A.	B.	C.	B′.	C′.
LiLi .... *ss*	$C\,^1\Pi_u$	269·7 **H**	5·771	2	2·93	2·539	2·999	2·941	3·000	2·965	−13·35	+2·36	+0·38	+2·39	+1·19
	$B\,^1\Sigma_u+$	253·2 **H**	..	..	3·11	2·593	3·099	3·004	3·100	3·027	−10·83	−0·35	−3·41	−0·32	−2·67
	$A\,^1\Sigma_g+$	**G** 351·6	..	..	2·67	2·325	2·678	2·692	2·625	2·714	−12·92	−1·72	+0·82	−1·69	+1·27
BeO .... *sp*	$^1\Sigma^*$	1370·8	9·498	8 (*c*)	1·358	1·359	1·345	1·358	1·345	1·358	+ 0·07	−0·96	0·00	−0·96	0·00
	$^1\Pi$	1127·8	..	..	1·468	1·451	1·437	1·450	1·437	1·450	− 1·16	− 2·12	−1·23	−2·12	−1·23
	$^1\Sigma_g$	**G** 1486·9	..	..	1·327	1·322	1·310	1·320	1·310	1·321	− 0·38	−1·28	−0·53	− 1·28	0·45
CC ..... *pp*	$d\,^1\Pi$	1832·5	9·894	8 (*s*)	1·251	1·226	1·220	1·232	1·221	1·231	− 2·00	− 2·48	−1·52	− 2·40	−1·60
	$B\,^3\Pi$	1792·6	..	..	1·261	1·234	1·228	1·241	1·228	1·241	− 2·14	− 2·62	− 1·59	− 2·62	−1·59
	$b\,^1\Pi$	1608·3	..	..	1·315	1·279	1·269	1·286	1·270	1·286	− 2·74	−3·50	− 2·21	−3·42	2·21
	$A\,^3\Pi_g$	**G** 1641·6	..	..	1·308	1·271	1·262	1·278	1·262	1·277	− 2·83	−3·52	−2·29	−3·52	−2·37
BeF .... *sp*	$A\,^2\Pi$	1172·6	10·07	9 (*c*)	1·390	1·418	1·404	1·403	1·404	1·403	+ 2·02	+1·01	+0·92	+ 1·01	+0·92
	$X\,^2\Sigma_g$	**G** 1265·6	..	..	1·357	1·382	1·368	1·367	1·368	1·368	+ 1·84	+0·81	+0·74	+0·81	+ 0·81
$B^{11}O$ ... *pp*	$B\,^2\Sigma$	1280·3	10·75	9 (*b*)	1·301	1·361	1·348	1·361	1·348	1·361	+ 4·61	+3·61	+4·61	+ 3·61	+4·61
	$A\,^2\Pi$	1259·1	..	..	1·348	1·370	1·355	1·368	1·355	1·369	+ 1·63	+0·52	+1·48	+0·52	+ 1·56
	$X\,^2\Sigma_g$	**G** 1885·4	..	..	1·199	1·197	1·196	1·197	1·196	1·196	− 0·17	−0·25	− 0·17	−0·25	−0·25

$B^{10}O$ ... pp	$X^2\Sigma_g$	G 1940·5	10·15	9 (*b*)	1·199	1·197	1·196	1·186	1·196	1·185	·	0·17	−0·25	- 1·08	−0·25	−1·17
CN .... pp	$B^2\Sigma$	2164·2	10·67	9 (*a*)	1·148	1·145	1·152	1·143	1·152	1·142	−	0·26	+ 0·35	−0·44	+0·35	−0·52
	$A^2\Pi$	1788·7	..	..	1·236	1·220	1·216	1·218	1·216	1·217	−	1·29	−1·62	−1·46	−1·62	−1·54
	$X^2\Sigma_g$	G 2068·8		..	1·169	1·162	1·166	1·160	1·166	1·160	−	0·60	−0·26	−0·77	- 0·26	−0·77
CO .... pp	$B^1\Sigma$	2182 H	11·31	10 (*b*)	1·118	1·131	1·140	1·120	1·140	1·119	+	1·16	+1·97	+0·18	+1·97	+0·09
	$A^1\Pi$	1516·7 H	..	..	1·232	1·277	1·266	1·264	1·266	1·264	+	3·65	+2·76	+2·60	+2·76	+ 2·60
NN .... pp	$B^3\Pi$	1732·8 H	11·54	10 (*s*)	1·201	1·217	1·213	1·209	1·213	1·209	+	1·33	+1·00	+0·67	+1·00	+0·67
	$A^3\Sigma$	1460·4 H	..	..	1·291	1·288	1·277	1·282	1·277	1·280	−	0·23	−1·08	−0·70	−1·08	−0·85
	$X^1\Sigma_g$	G 2359·6 H	..	..	1·094	1·097	1·114	1·092	1·114	1·091	+	0·27	+1·83	−0·18	+1·83	−0·27
NO .... pp	$B^2\Pi$	1037·6	12·31	11 (*a*)	1·413	1·428	1·414	1·413	1·414	1·411	+	1·06	+0·07	0·00	+0·07	−0·14
	$A^2\Sigma$	2375·6	..	..	1·060	1·083	1·104	1·072	1·105	1·071	+	2·17	+4·15	+1·13	+ 4·25	+1·04
	$X^2\Pi_g$	G 1906·5	..		1·146	1·166	1·169	1·154	1·170	1·153	+	1·75	+2·01	+0·70	+2·09	+0·61
OO .... pp	$B^3\Sigma$	710·1	13·19	12 (*s*)	1·599	1·603	1·604	1·579	1·604	1·579	+	0·25	+0·31	−1·25	+0·31	−1·25
	$A^1\Sigma$	1432·6	..	..	1·223	1·268	1·259	1·251	1·259	1·249	+	3·68	+ 2·94	+2·29	+2·94	+ 2·13
	$X^3\Sigma_g$	G 1584·	..	..	1·204	1·226	1·221	1·209	1·221	1·209	+	1·83	+ 1·41	+0·42	+1·41	+0·42
FF .... pp	$^1\Pi$	1139·8 H	15·67	14	1·278	1·329	1·316	1·315	1·316	1·314		+3·99	+ 2·97	+2·90	+2·97	+2·81
Average percentage errors (neglecting sign)												**2·66**	**1·68**	**1·25**	**1·68**	**1·26**

being: A 2·66, B 1·68, C 1·25. The use of the modified B′ and C′ formulæ hardly affects the average errors: B′ 1·68, C′ 1·26. The C and C′ formulæ are thus the best in this period.

Some factor would appear to enter in, making the period constants of the A formula more structurally-sensitive (to *s* and *p* atoms), whilst the B and C formulæ become more sensitive to electron configurations in the later periods.

IV. *Results in the* KK^+ *Period.*

Table II. reveals that similar considerations hold in the KK^+ as in the KK period. The average errors are: A 2·57, B 2·85, C 1·20, B′ 2·87, C′ 1·32 (± per cent.). Here the C formula again gives the best results, and no improvement is effected by using B′ and C′ instead of B and C respectively. It may appear that some improvement might be brought about in the A and B formulæ by using different period constants from those of the neutral di-atoms of the KK period.

V. *Results in the* KL *and* KM *Periods.*

The four formulæ A, B, B′, and C appear about equally good in the KL period (Table III.), the average errors running near together in the neighbourhood of 2·1 per cent. Allen and Longair[12] report that "the errors are slightly in favour of Douglas Clark's formula."

In the KM period (Table IV.), the C and C′ formulæ are the best, so far as may be argued from the consideration of only three electronic states for which data are available.

Tables I. to IV. deal with the KK, KK^+, KL and KM periods, that is, with di-atoms containing at least one atom of the K period. In these cases, the results show generally that the B′ and C′ modifications offer no sensible improvements on the B and C formulæ respectively. Hence the constants C_{ij} and k_{qr} are not needed for these periods: the modification of the constants is useful, however, for the LL and later periods (see below). It appears that C_{ij} and k_{qr} allow for some structurally-sensitive property of the period constants, but that the effect is not appreciable so long as the simplest types of di-atoms only are taken into account.

TABLE II.

Calculated Results in the KK^+ Period.

$K=(10.602)$; $d_{ij}=d'_{ij}=(0\cdot680)$; $C_{ij}=(0\cdot186)$; $k=9.700$: $k'=650$: $k_{pp}=9,680$; $k'_{pp}=780$.

Di-atom.	State.	ω_e.	μ.	n.	r_e (expt.).	r_e (calc.). A.	B.	C.	B′.	C′.	Percentage errors. A.	B.	C.	B′.	C′.
CO^+ ... PP	$B\,^2\Sigma^+$	1722·1	11·31	9	1·16	1·223	1·219	1·206	1·219	1·190	−5·43	−5·09	−3·97	−5·09	−2·59
	$A\,^2\Pi_{inv}$	1564·5	..	..	1·24	1·263	1·254	1·245	1·254	1·228	−1·86	−1·13	−0·40	−1·13	0·97
	G $X\,^2\Sigma^-$	2212	..	..	1·11	1·126	1·136	1·100	1·136	1·094	−1·44	−2·34	−0·09	−2·34	−1·44
N_2^+ ... PP	$B\,^2\Sigma^+$	2417·7	11·54	9	1·071	1·089	1·107	1·076	1·107	1·071	−1·68	−3·36	−0·47	−3·36	0·00
	G $X\,^2\Sigma_g$	2206·8	..	..	1·113	1·123	1·123	1·110	1·134	1·102	−0·90	−1·80	−0·27	−1·89	−0·99
O_2^+ ... PP	$A\,^2\Pi_u$	898·9 H	13·19	11	1·41	1·481	1·469	1·447	1·469	1·440	−5·04	−4·19	−2·62	−4·19	−2·13
	G $X\,^2\Sigma$	1876·4 H	..	..	1·14	1·159	1·163	1·133	1·164	1·127	−1·67	−2·02	−0·61	−2·11	−1·14
Average percentage errors (neglecting sign)											2·57	2·85	1·20	2·87	1·32

TABLE III.—Calculated Results in the KL Period.

$K=16{,}844$; $d_{ij}=0{\cdot}900$; $d'_i=0{\cdot}940$; $C_{ij}=0{\cdot}153$; $k_{pp}=12{,}850=k$; $k'=k'_{pp}=0$.

Di atom.	State.	ω_e.	μ.	n.	r_e (expt.).	r_e (calc.). A.	B.	C.	B′.	Percentage errors. A.	B.	C.	B′.
AlO *pp*	$B\,^2\Sigma^+$	868·2	16·56	9 (*b*)	1·663	1·683	1·707	1·702	1·696	+1·20	+2·65	+2·35	+1·99
	G $X\,^2\Sigma^+$	977·0	..	..	1·614	1·617	1·646	1·636	1·639	+0·19	+1·98	+1·36	+1·55
CP *pp*	$^2\Pi$	1062	14·27	9 (*a*)	1·667	1·614	1·588	1·592	1·522	−3·18	−4·74	−4·45	−8·70
	$B\,^2\Sigma^*$	836·3	..	..	1·685	1·747	1·707	1·723	1·696	+3·68	+1·31	+2·25	+0·65
	G $X\,^2\Sigma$	1239·7	..	..	1·558	1·532	1·520	1·512	1·585	−1·67	−2·44	−2·95	+1·73
SiN *pp*	$^2\Sigma^*$	1034·4	15·39	9 (*a*)	1·576	1·607	1·583	1·605	1·580	+1·97	+0·44	+1·84	+0·25
	G $X\,^2\Sigma^+$	1151·7	..	..	1·568	1·550	1·535	1·549	1·536	−1·15	−2·15	−1·21	−2·04
CS *pp*	G $X\,^1\Sigma$	1282·5	14·40	10 (*b*)	1·56	1·512	1·505	1·469	1·507	−3·08	−3·53	−5·83	−3·40
SiO *pp*	$^1\Sigma$	1240·5 H	16·80	10 (*b*)	1·505	1·490	1·487	1·485	1·491	−0·10	−1·20	−1·33	−0·93
PN *pp*	$^1\Pi$	1103·1	16·04	10 (*s*)	1·542	1·563	1·545	1·545	1·545	−1·36	−0·20	−0·20	+0·19
	G $^1\Sigma$	1337·2	..	..	1·487	1·465	1·467	1·448	1·472	−1·48	−1·35	−2·62	−1·01
PO *pp*	$^2\Sigma$	1230·6	17·41	11 (*a*)	1·446	1·486	1·484	1·466	1·487	+2·77	+2·63	+1·38	+2·84
SO *pp*	$B\,^3\Sigma$	628·7 H	17·6	12 (*s*)	1·769	1·856	1·818	1·807	1·793	+4·92	+2·77	+2·15	+1·36
	G $X\,^3\Sigma$	1123·7	..	..	1·489	1·529	1·518	1·489	1·519	+2·69	+1·95	0·00	+2·01
Average percentage errors (neglecting sign)										**2·10**	**2·10**	**2·13**	**2·05**

Table IV.

Calculated Results in the KM Period.

$K=20{,}150$; $d_{ij}=0{\cdot}985$; $d'_{ij}=1{\cdot}060$; $C_{ij}=0{\cdot}125$; $k=13{,}310$; $k_{sp}=13{,}670$; $k_{p'l}=13{,}130$; $k'=k'_{sp}=k'_{p\bar{a}}=0$.

Di-atom.	State.	ω_e.	μ	n.	r_e (expt.).	r_e (calc.). A.	B.	C.	B′.	C′.	Percentage errors. A.	B.	C.	B′.	C′.
CaF sp	$^2\Sigma$	586·7 **H**	21·99	9 (*c*)	(1·98)	1·942	1·87	1·963	1·84	1·981	−1·92	−5·55	−0·86	−7·32	+0·05
TiO $pd\ \frac{1}{4}$	$X\,^3\Pi$ **G**	1008·1	19·78	10(*b*)	1·617	1·650	1·624	1·610	1·620	1·603	+2·04	+0·43	−0·43	+0·19	−0·87
	$C\,^3\Pi$	837·9	..	..	1·690	1·756	1·708	1·712	1·693	1·705	+3·90	+1·07	+1·30	+0·18	+0·89
	Average percentage errors (neglecting sign)										**2·62**	**2·35**	**0·86**	**2·56**	**0·60**

The general conclusion is reached that *the C formula is either better than, or at least equal to, the A, B, and B′ modifications for di-atoms containing one atom having no more than one complete K ring*, and that it may therefore be recommended to spectroscopists in practical use as the best approximation in these cases which has up to the present time been discovered. The reason for its efficiency seems to be that allowance is made for *both* period and group in each case.

VI. *Results in the* LL *Period.*

In Table V., application of both the B and C formulæ, sets NaNa in a class by itself, as compared with PP, SS and ClCl. The average errors for the three states of NaNa are :—B 9·83, C 18·60 per cent. Since this is the only *ss*-di-atom in the Table, it seems necessary to suppose that the period constants of the B and C formulæ are structurally-sensitive in more complex di-atoms, and that Badger's modified B′ formula meets this difficulty by introduction of the second period constant C_{ij}. It might perhaps have been equally well met by using different d_{ij}'s for *ss*- and *pp*-di-atoms ; this is the procedure of the present authors, who adopt different constants k_{ss}= 7,150 and k_{pp}=14,250. The enormous discontinuity between the two numbers, the second being about twice the first, suggests that this method of meeting the difficulty is sound. The modified formulæ then give average errors for the period which are reduced to :—B′ 2·55, C′ 2·66 per cent., the results being about equally good. The results of calculation using the C formula are not included in Table V., for it is impossible to find a suitable value of k. If, provisionally, k is set equal to k_{pp}, then the average error for the period is as large as 8·62 per cent. It does not appear that a correction of the above type would suit the A formula, since the error of one level of NaNa in calculating r_e is negative, as against larger positive errors for the two other states.

VII. *Results in the* MM, $\overline{\text{MM}}$, $\overline{\overline{\text{NN}}}$ *Periods.*

The data available for the construction of Tables VI., VII., and VIII. are so meagre that it is extremely difficult to draw definite conclusions from the results. It will be noted that k_{pp} in the M$\overline{\text{M}}$ period is about twice as great

Table V.

Calculated Results in the LL Period.

K=23,253 ; d_{ij}=1·180 ; d_{ij}=1·250 ; C_{ij}=0·118 ; k_{ss}=7,150 ; k_{pp}=14,250 ; $k'=k'_{ss}=k'_{pp}=0$.

Di-atom.	State.	ω_e.	μ.	n.	r_e (expt.).	r_e (calc.). A.	B.	B′.	C′	Percentage errors. A.	B.	B′.	C′.
NaNa ... *ss*	$A\,^1\Sigma_g^+$	**G** 159·23 **H**	18·96	2	3·070	3·225	3·397	3·153	3·166	+5·05	+10·65	+2·70	+3·13
	$B\,^1\Sigma_u^+$	115·60 **H**	..	..	3·663	3·588	3·925	3·606	3·523	−2·05	+ 7·15	−1·56	−3·82
	$C\,^1\Pi_u$	123·79 **H**	..		3·404	3·508	3·802	3·501	3·443	+3·06	+11·69	+2·85	+1·15
PP .... *pp*	$X\,^1\Sigma_g^+$	**G** 779·3 **H**	25·58	10 (*s*)	1·88	1·803	1·876	1·848	1·795	− 4·10	− 0·21	−1·70	− 4·52
	$a\,^1\Sigma_g^+$	470·95 **H**	..	..	2·12	2·137	2·154	2·086	2·122	+ 0·80	+ 1·60	−1·60	+0·09
SS ..... *pp*	$X\,^3\Sigma_g^-$	**G** 727·4 **H**	26·45	12 (*s*)	1·73	1·839	1·801	2·029	1·781	+6·30	+4·11	+8·04	+2·95
ClCl .... *pp*	$^1\Sigma_g^+$	**G** 564·9 **H**	28·86	14	1·983	1·972	2·009	1·961	1·889	−0·55	+1·31	−1·11	−4·74
	0_u^+	261·1 **H**	..	..	2·42	2·550	2·567	2·440	2·443	+5·37	+6·07	+0·83	+0·95
	Average percentage errors (neglecting sign) *									**3·41**	**5·35**	**2·55**	**2·66**

* Errors omitting NaNa : A 3·42, B 2·66, B′ 2·55, C 2·65 per cent.

Table VI.

Calculated Results in the MM Period.

$K=31{,}280$; $d_{ij}=(1{\cdot}350)$; $d'_{ij}=1{\cdot}480$; $C_{ij}=0{\cdot}129$; $k_{ss}=7{,}900$; $k_{ss}=0$.

Di-atom.	State.	ω_e.	μ.	n.	r_e (expt.).	r_e (calc.). A.	B.	B′.	C′.	Percentage errors. A.	B.	B′.	C′.
KK....	$C\,^1\Pi_u$	75·00 **H**	32·15	2	4·22	4·190	4·421	4·197	4·207	−0·71	+4·76	−0·55	−0·31
$_{ss}$	$A\,^1\Sigma^+_g$	**G** 92·64 **H**	..	..	3·91	3·904	4·018	3·84	3·921	− 0·15	+2·76	− 1·79	+0·28
	Average percentage errors (neglecting sign)									**0·43**	**3·76**	**1·17**	**0·29**

Table VII.

Calculated Results in the M$\bar{\text{M}}$ Period.

$K=(31{,}280)$; $d_{ij}=1{\cdot}350$; $d'_{ij}=(1{\cdot}480)$; $C_{ij}=(0{\cdot}129)$; $k_{pp}=14{,}360$; $k'_{pp}=0$.

Di atom.	State.	ω_e.	μ.	n.	r_e (expt.).	r_e (calc.). A.	B.	B′.	C′.	Percentage errors. A.	B.	B′.	C′.
BrBr ...	$B0^+_u$	165·39 **H**	65·89	14	2·65	2·856	2·777	2·743	2·852	+7·78	+ 4·79	+ 3·51	+7·62
$_{pp}$	$X\,^1\Sigma^+_g$	**G** 323·86 **H**	..	..	2·28	2·282	2·262	2·287	2·280	+0·09	−0·79	+0·31	0·00
	Average percentage errors (neglecting sign)									**3·93**	**2·79**	**1·91**	**3·81**

TABLE VIII.

Calculated Results in the NN Period.

$K=45{,}500$; $d_{ij}=1{\cdot}635$; $d'_{ij}=1{\cdot}760$; $C_{ij}=0{\cdot}118$; $k_{pp}=16{,}090$; $k'_{pp}=0$.

Di-atom.	State.	ω_e.	μ.	n.	r_e (expt.).	r_e (calc.). A.	B.	B′.	C′.	Percentage errors. A.	B.	B′.	C′.
II $_{pp}$	$B0^-_u$	128·0 H	104·65	14	3·01	3·263	4·762	4·444	3·228	+8·41	−58·2	−47·6	−7·24
	$X\,^1\Sigma^-_g$	G 214·26 H	..	..	2·66	2·749	3·853	3·664	2·717	+3·35	+44·8	−37·7	+2·14
TeTe ... $_{pp}$	$^3\Sigma$	G 250·9 H	105·1	12 (*s*)	2·85	2·606	3·629	3·471	2·644	−8·56	−27·3	−21·8	−7·20
	Average percentage errors (neglecting sign)									**6·77**	**43·4**	**35·7**	**5·53**

as k_{ss} in the MM period, resembling the state of affairs noted above in the LL period. So far as may be judged in these periods, the B′ formula produces better results than B, whilst the C′ formula continues to hold its own reasonably well. The assigned constants for B and B′ formulæ seem to be too large in the $\bar{N}\bar{N}$ period.

It seems, therefore, that the C (or C′) formula is generally satisfactory for all di-atoms containing one atom of the K period, whilst for more complex cases the C′ modification may be applied.

VIII. *General Method and Discussion.*

We next collate the constants used, in Table IX.

TABLE IX.

Constants of Modified Morse Formulæ.

Formula.	A.	B.	B′.		C.	C′.			
Constant. / Period.	K.	d_{ij}.	d'_{ij}.	C_{ij}.	$k-k'$.	$k-k'$.			
						ss.	*sp.*	*pp.*	*pd.*
KK^+....	(10,602)	(0·680)	(0·680)	(0·186)	8,900	..	..	8,900	..
KK	10,602	0·680	0·680	0·186	9,700	9,940	9,710	9,680	..
KL	16,844	0·900	0·940	0·153	12,850	..	..	12,850	..
KM	20,150	0·985	1·060	0·125	13,310	..	13,670	..	13,13
LL	23,253	1·180	1·250	0·118	..	7,150	..	14,250	..
MM	31,280	(1·350)	1·480	0·129	..	7,900	..	..	..
$\bar{M}\bar{M}$	(31,280)	1·350	(1·480)	(0·129)	..	..	..	14,360	..
NN	45,500	1·635	1·760	0·118	..	..	..	16,090	..

The constants tend to increase steadily from the KK^+ to the $\bar{N}\bar{N}$ period, except in the cases of C_{ij} and $(k-k')_{ss}$. The numbers given in brackets are taken from other periods : thus, in the absence of an allowance for ionization, the constants for KK^+ are taken as the same as the corresponding numbers for KK. Similarly, the constants for $\bar{M}M$ may be taken from MM, and *vice versa*. The progression of the numbers suggests that possibly

better agreement might be obtained using the A and B formulæ if different constants were assigned to KK^+ from those of KK, and of MM from those of MM.

The average deviations from experiment using the five formulæ are next considered, as set forth in Table X. The four simpler periods containing one atom of the K period are treated, and the mean errors are calculated (*a*) by the arithmetic mean, (*b*) by the " weighted " mean, taking into account the number of states corresponding to any given number. Judged either way, the C formula

TABLE X.

Analysis of Errors for Di-atoms containing at least One Atom of the K Period.

		Average percentage errors (± per cent.).									
Period.	No. of states.	A.	B.	B′.	C.	C′.					
						ss.	*sp.*	*pp.*	*pd.*	Mean (*a*).	Mean (*b*).
KK^+..	7	2·57	2·85	2·87	1·20	..	..	1·32	..	1·32	1·32
KK ..	31	2·66	1·68	1·68	1·25	1·71	0·68	1·35	..	1·25	1·27
KL ..	14	2·10	2·10	2·05	2·13	..	..	2·13	..	2·13	2·13
KM ..	3	2·62	2·35	2·56	0·86	..	0·05	..	0·88	0·47	0·60
Mean errors (*a*)		2·49	2·25	2·29	1·36	1·71	0·37	1·60	0·88	1·12 1·29	..
Mean errors (weighted) (*b*)		2·50	1·97	1·97	1·44	1·71	0·58	1·59	0·88	..	1·46

is the best, and B′, C′ give no sensible improvement on B, C respectively. 55 electronic states of 25 di-atoms are covered in this analysis.

Table XI. contains the summarized results for LL and higher periods. 15 states of 8 di-atoms are included, the means (*a*) and (*b*) of the results by the five formulæ being shown. The errors seem generally larger than in Table X., but doubtless the experimental uncertainty tends to be greater: besides which, the number of states is too few for satisfactory generalizations. Judging

by the available data, B′ and C′ are now better than B and C respectively, and C′ gives the best results.

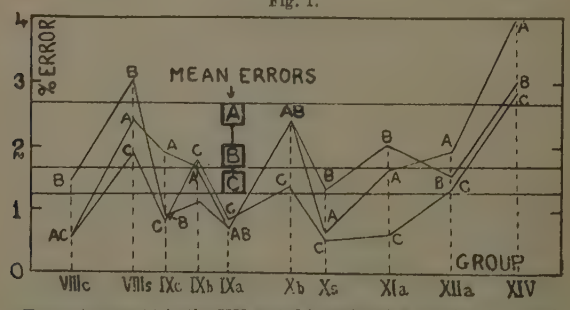
TABLE XI.

Analysis of Errors for Di-atoms of LL and Higher Periods.

Period.	No. of states.	Average percentage errors (± per cent.).							
		A.	B.	B′.	C.	C′.			
						ss.	*pp.*	Mean (*a*)	Mean (*b*)
LL	8	3·41	5·35	2·55	(8·62)*	2·70	2·65	2·68	2·67
MM	2	0·43	3·76	1·17	0·29	0·29	..	0·29	0·29
MM̄	2	3·93	2·79	1·91	3·81	..	3·81	3·81	3·81
NN	3	6·77	43·4	35·7	5·53	..	5·53	5·53	5·53
Mean errors (*a*)		3·64	13·83	10·33	4·56	1·50	4·00	2·75 3·08	..
Mean errors (weighted) (*b*)		3·75	12·41	8·91	6·24	1·74	3·75	..	3·08

* Using k_{pp}.

Fig. 1.



Errors (per cent.) in the KK period in estimating r_e using A, B, and C formulæ.

The weighted means of the errors for 70 electronic states of 33 di-atoms are: A 2·77, B 4·21, B′ 3·46, C 2·47, C′ 1·81 (± per cent.).

Fig. 1 shows the mean errors in the KK period in different groups. The C line tends generally to run below the A and B lines, whilst the pairs B, B′, and C, C′ are indistinguishable on the scale chosen. The mean errors for the period are shown by three lines parallel to the group axis. If we exclude LiLi in taking the mean by the A formula, the A practically coincides with the B mean line, so that C still gives the best results.

The result of this analysis shows that *spectroscopists may use the C formula in forming estimates of r_e from experimental ω_e's for di-atoms containing one atom of the K period. For more complex di-atoms, the C′ formula may be used.* For this purpose, the di-atom should be assigned to its class (*ss*, *sp*, *pp*, or *sd*), the period constants being taken from Table IX. As further experimental work is carried out, it should become possible to extend the available constants, by filling in the spaces in the table. To facilitate the assignment of di-atoms to their classes, a small periodic table is presented in Table XII., with each atom appropriately assigned to its class according to the method explained in the present paper.

It may appear remarkable that such apparently different types of formulæ should reproduce the experimental internuclear distances with such good general agreement. It is frequently noticeable that the errors using different formulæ for a given electronic state tend to be all positive or all negative, and when one is large the others tend to be large also. It seems difficult to escape the conclusion that some hitherto undiscovered general law is involved, and that the various constants used by different workers must be capable of expression in terms of other fundamental molecular constants, and will also be found to be related amongst themselves. It seems generally true that the A formula is more successful for more complex di-atoms than for simple, the converse tending to hold for the B and C formulæ. It might be anticipated, therefore, that a general law may be discovered which will reduce to the B or C forms for simpler cases and to the A form for the more complex. With respect to relations between the constants, it is clear, for example, that if equations (A 1) and (C 1) in the text hold, then the Group number n should be proportional to the reduced mass μ in a given period. This may be tested, when it is found that the proportionality

might be described as a tendency rather than an exact relationship.

The fact that Morse's Function has been shown to be periodic in the di-atomic groups may have some bearing on the present suggestions for correction. The use of $n^{\frac{1}{2}}$ and $\mu^{\frac{1}{2}}$, for example, in the C and A formulæ respec-

TABLE XII.

Classification of Atoms into *s*-, *p*-, and *d*-types.

Type.	Atom.									Group.
s	Li	Na	K	Cu	Rb	Ag	Cs	Au	87	I.
	Be	Mg	Ca	Zn	Sr	Cd	Ba	Hg	Ra	II.
p	B	Al	..	Ga	..	In	..	Tl	..	III.
d	..	..	Sc	..	Y	..	La	..	Ac	
p	C	Si	..	Ge	..	Sn	..	Pb	..	IV.
d	..	..	Ti	..	Zr	..	Hf	..	Th	
p	N	P	..	As	..	Sb	..	Bi	..	V.
d	..	..	V	..	Cb	..	Ta	..	Pa	
p	O	S	..	Se	..	Te	..	Po	..	VI.
d	..	..	Cr	..	Mo	..	W	..	U	
p	F	Cl	..	Br	..	I	..	85	..	VII.
d	..	..	Mn	..	Ma	..	Re	..	..	
d	..	..	Fe	..	Ru	..	Os	..	..	VIII.
	..	..	Co	..	Rh	..	Ir	..	..	
	..	..	Ni	..	Pd	..	Pt	..	..	
Period	K	L	M	M	N	N	O	O	P	

tively, might be regarded as a consequence of some Principle of "Neutralized Periodicity," where, by combining two functions which are periodic in opposite senses, compensation is brought about, the resultant function being approximately constant in each period.

The measure of success attained by the C formula may perhaps be partly accounted for by the fact that it is the only modification of Morse's Rule which makes allowance for *both the period and the group of a di-atom simultaneously*,

besides introducing *a correction for molecule-ions*. Further, it seems clear from the present analysis that the modified expression should take account, not only of *numbers* of valency electrons of the constituent atoms, but also of their *quantum species* (s, p, d . . .), especially in the higher di-atomic periods. It may be expected that some formula may eventually be found having theoretical background which will in some way embody the principles upon which the natural periodic group classification of non-hydride di-atoms is based.

IX. *Summary.*

Formulæ suggested by Allen and Longair, Badger, and Douglas Clark connecting fundamental vibration frequency and internuclear distance have been tested in respect of their capacity to reproduce experimental internuclear distances for 70 ground and excited electronic states of 33 non-hydride di-atoms. For di-atoms containing one atom of the K period, Douglas Clark's formula ($\omega_e r_e^3 n^{\frac{1}{2}} = k - k'$) gives the best results, whilst Badger's recent modification effects no improvement on his original formula. More complex di-atoms require attention to be paid to the electron configurations of the atoms concerned in assigning the period constants, and Douglas Clark's formula ($\omega_e r_e^3 n^{\frac{1}{2}} = k_{q'}$) is generally more satisfactory than the other proposals. This formula may therefore be recommended for use in band spectroscopy, following the procedure explained in the paper.

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University of Leeds.
June 3rd, 1936.

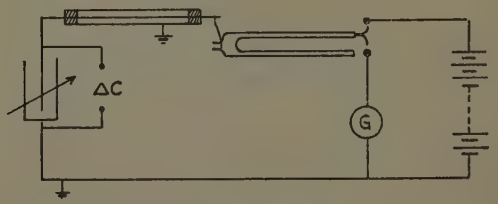
XCVIII. *Some Uses of a Calibrated Variable Condenser.* *By* L. G. H. Huxley, *M.A., D.Phil., F.Inst.P., Lecturer in Physics, University College, Leicester, and* K. N. Middleton, *Student, University College, Leicester* *.

Introduction.

SMALL capacities and related electrical quantities may conveniently be measured with inexpensive apparatus by the simple substitution method described below.

In this method a screened variable condenser which has been calibrated against a standard guard-ring cylindrical condenser is included, with an electrically maintained tuning-fork or a commutator, in the circuit shown in fig. 1, and is charged and discharged regularly

Fig. 1.



through the sensitive galvanometer G. The tuning-fork, however, was actually placed in the earth lead to the condenser in order to reduce external capacities.

When the galvanometer deflexions θ were plotted as ordinates against the capacity C of the variable condenser an accurately linear relationship was found to hold between these quantities, as may be seen from the experimental curve AB in fig. 2, in which the relevant experimental data are also given.

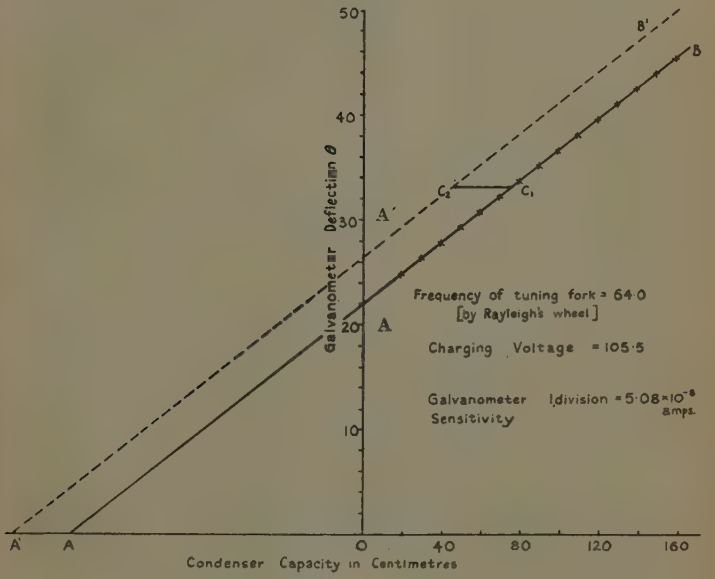
This agreement serves as a satisfactory check on the calibration of the variable condenser.

The line AB does not pass through the origin, and the intercept OA on the axis of abscissæ represents the value of the external capacity due to the leads, which is not included in the capacity C indicated on the dial of the variable condenser. When, therefore, a further external

* Communicated by the Authors.

capacity ΔC is added in parallel with those of the condenser and leads, of amount corresponding to AA′, the line AB is displaced to A′B′ without change of slope. Consequently when the lines AB without the additional capacity ΔC and AB with ΔC have been found by experiment the additional capacity is represented by the displacement AA′ of the intercept. It is unnecessary, however, to find the line A′B′ in order to obtain the value of ΔC,

Fig. 2.



as the latter may be read immediately from the scale of the variable condenser, as that change which must be made in its capacity in order to reduce the galvanometer deflexion to the value observed before the addition of ΔC. This operation is equivalent to passing from any point C_1 on the line AB to the point C_2 on A′B′ with the same ordinate. It is clear that $\Delta C(=AA')$ is the difference (C_1-C_2) of the abscissæ of these points.

As this procedure is merely one of simple substitution, in which the external leads are left undisturbed, it is

applicable to other circuits, such as Maxwell's bridge with the tuning-fork and condenser in one arm, or to the bridge circuit for the comparison of small capacities mentioned in the next paragraph.

Construction and Calibration of the Variable Condenser.

The standard variable condenser used in these measurements was made by mounting a ·0005 μF (or smaller) variable condenser of good quality rigidly within a screening case of metal. A flat-bottomed aluminium saucepan was found to make a suitable case. The lid was mounted inverted on three ebonite pillars on a baseboard, while the bowl, which could then be fitted to the lid, carried the variable condenser within it and mounted to its bottom. The position of the set of movable plates of the condenser, which are put into electrical contact with the bowl, was given by a scale pasted to the inverted flat base of the saucepan and a pointer. Electrical connexion to the inner fixed plates of the condenser was made by a metal rod passing through one of the rivet-holes where the handle had previously been attached.

The movable plates and the case constitute one conductor of the system and the fixed plates the other. In order to find the coefficients of capacity $C(=C_{12}=C_{21})$ for the various positions of the pointer the system was calibrated against a standard guard-ring cylindrical condenser by use of a well-known null method due to Townsend. In this method the two condensers to be compared are included in a bridge circuit with two resistances, an electrometer and a battery *.

Of two condensers so constructed and calibrated the one had a capacity range of 17–460 centimetres, and the other, in which alternate plates had been removed, a range of 12–85 centimetres. In the latter the scale could be read with an accuracy of 0·1 cm. at its lower end and 0·2 cm. at its upper.

Experimental Check on Method of Measuring Small Capacities.

As an experimental check on the method it was applied to measure the capacity per unit length of a pair of

* Pidduck, 'Treatise on Electricity,' 1st ed. 1916. p. 143 ; 2nd ed. p. 136.

cylindrical parallel rods. The rods were constructed to telescope within a pair of tubes (fig. 3 *a*), and the capacity ΔC added to the system when the rods were withdrawn vertically a distance l was found experimentally by the method already described. In this way end effects are eliminated, and the measured capacity ΔC may be compared with the theoretical value appropriate to a length l of a pair of similar conductors but infinite in length. The following results show a satisfactory agreement between the measured and calculated capacities: $l=20$ cm.; diameters of rods, $2a=0{\cdot}48$ cm.; distance between axes of rods, $d=2{\cdot}22$ cm.

Theoretical capacity of a length l of the system

$$= \frac{l}{4 \cosh^{-1} d/2a} = 0{\cdot}113\, l = 2{\cdot}26 \text{ cm.}$$

Measured capacity $\Delta C = 2{\cdot}30$ cm. $= 0{\cdot}115\, l$ cm.

The experimental value was independent of the charging voltage, of the initial setting of the variable condenser, and of the frequency of the fork employed.

The capacity of a Ferranti 0–450 volt electrostatic voltmeter was found to be 30 cm. in the position of zero deflexion.

It is also possible to demonstrate with fair accuracy that the capacity of an isolated sphere is numerically equal to its radius by placing the sphere on a tall ebonite rod which carries a fine connecting wire to the inner plate of the condenser. Earthed metal sheets should surround the sphere at some distance from it.

To obtain trustworthy measurements of small capacities in these experiments it is advisable to charge the inner electrode of the condenser and to stand the case on an earthed metal sheet to which it is connected. Leads to the inner electrode should be supported within earthed metal tubes to give them a fixed capacity in parallel with that of the condenser.

Measurement of the Ratio of the Electromagnetic to the Electrostatic Unit of Charge.

The ratio c of the electromagnetic to the electrostatic unit of charge is easily obtained from the slope of the line AB (fig. 2).

If C_1 and C_2 are two values of the condenser capacity and i_1 and i_2 the corresponding galvanometer currents

in electromagnetic units then $(C_1-C_2)nV=(i_1-i_2)$, from which the capacity interval, already known in electrostatic units, may be found in electromagnetic units.

The mean value of c which is found from the line AB (fig. 2) and the accompanying data is $c=2{\cdot}99\,.\,10^{10}$. This number provides a further satisfactory check on the reliability of the calibration of the condenser.

This modification of the usual method for obtaining c is suitable as a class experiment for honours students, since it is more accurate and reliable than the procedures usually recommended in text-books of experimental physics.

Measurement of Dielectric Constant.

The dielectric constant of a liquid may be obtained rapidly with the apparatus shown in fig. 3 (*b*).

A is a copper tube 32 cm. long and of 2·34 cm. internal diameter, and B a metal rod or tube of external diameter 1·28 cm. which is fitted into A and supported coaxially by insulating rings, the lower of which is shaped to permit the flow of liquid past it when B is inserted into A. M is a gauge which gives the level of the liquid within A. The dielectric constant of the liquid is determined by measuring the change in capacity ΔC which results from running out a length l of the liquid from the tap T.

If C′ is the calculated capacity per unit length of the condenser AB (considered infinite in length) and K the dielectric constant of the liquid, then, since end effects are eliminated,

$$\Delta C=(K-I)lC'.$$

A number of organic liquids were examined, and the values found for their dielectric constants were independent of the charging voltage, the dimensions of the inner electrode, and of the frequency of the tuning-fork used. The method fails, however, unless the liquid is highly insulating. For liquids with a slight conductivity it would be necessary to adapt the method for use with high-frequency oscillations. The following results are typical :

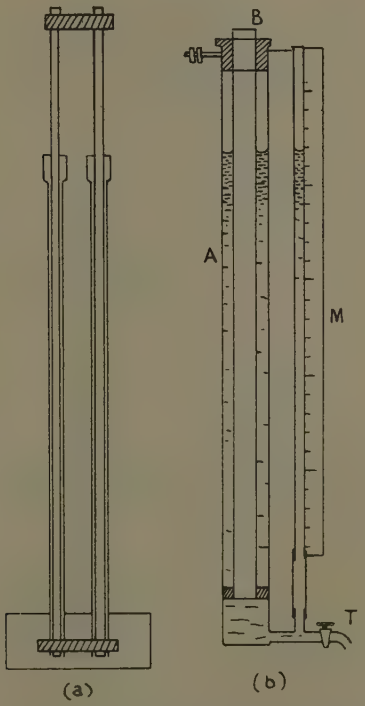
$$C'=0{\cdot}828 \text{ cm.}\,;\ l=25 \text{ cm.}$$

With xylene (commercial xylol) $\Delta C=27{\cdot}5$ cm., so that $K=2{\cdot}33$; with carbon-tetrachloride $\Delta C=25{\cdot}0$ cm., giving $K=2{\cdot}23$.

[The results obtained with the following liquids were : paraffin oil K=2·12 ; chlorobenzene K=5·8 ; hexane K=1·87. In all instances the values found were in good agreement with the best accepted values.]

By an analogous method the dielectric constant of solid paraffin wax was found to be 2·1.

Fig. 3.



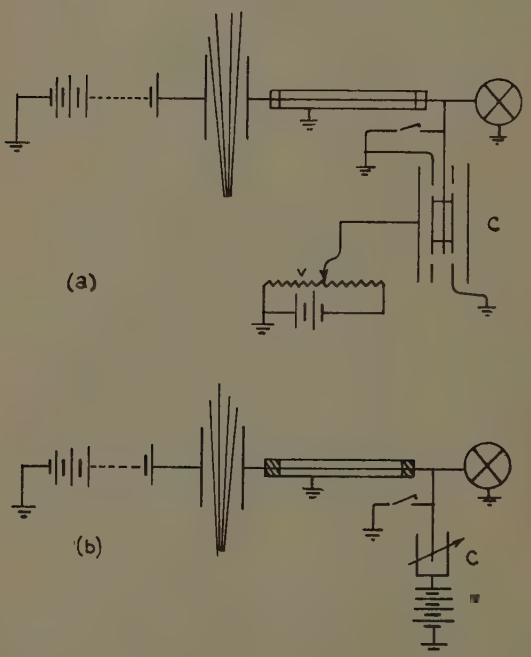
Measurement of Small Currents.

The variable condensers used in these experiments can also be usefully employed in a convenient modification of Townsend's induction balance for measuring small photoelectric and ionization currents *.

* J. S. Townsend, Phil. Mag. vi. p. 143 (1903); 'Electricity in Gases,' p. 20.

In fig. 4 (*a*) the well-known Townsend arrangement is shown. The electrometer system is maintained at zero potential, although receiving charge, by alteration of the potential v tapped from the potentiometer. The charge acquired by the insulated system is accurately

Fig. 4.



$Q=Cv$, in which C is the capacity of the standard guard-ring condenser.

In the modified scheme (fig. 4 (*b*)), the deflexion of the electrometer is prevented by changing the capacity of the calibrated variable condenser from C_1 to C_2 as charge is acquired by the insulated system. The charge Q is in this case given by $Q=v(C_1-C_2)$, in which v is constant.

The method has advantages of continuity of variation,

compactness, flexibility, and accuracy, and has been used with success by honours students in this laboratroy during some two years.

The other methods described in this paper have also proved suitable for requirements of honours students.

XCIX. *The Computation of the Strength of Windows.*

By A. F. Dufton, *M.A., D.I.C.**

1. In an earlier communication † nomograms were described for computing the strength of windows from the formula

$$p=2\,ft^2(a^2+b^2)/ka^2b^2, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where p is the uniform pressure which the panes will bear, a the length, b the breadth, and t the thickness of the pane, f the modulus of rupture, and where k, the factor of safety, includes also a factor which, for glass, has the value of about 1·2.

2. It would seem desirable to employ in the place of equation (1) the more accurate formula

$$4ft^2=3p\left[1-\frac{5}{6}\frac{a^2b^2}{a^4+b^4}\right]\left[\frac{a^4b^2}{a^4+b^4}\right], \quad . \quad . \quad (2)$$

which emerges from the approximate analysis by Dr. Marcus ‡, and to use, moreover, a factor of safety which will provide for a definite risk of failure.

3. In fig. 1 the distributions of the moduli of rupture of a number of specimens, both of plate glass and of sheet glass, are plotted on permille paper.

The specimens, which had a length-to-breadth ratio of 1·6 and were bedded on putty, were broken under uniformly distributed pressure by evacuating the air from below the frame supporting their edges. It will be observed that in the case of plate glass the distribution is normal and that sheet glass is approximately twice as strong as plate glass of the same thickness.

It is clear from the figures that if a modulus of rupture of 2000 lb./sq. in. is assumed for the particular batch of

* Communicated by the Author.

† Phil. Mag. xvii. p. 792 (1934).

‡ 'Die Theorie Elastischer Gewebe,' Julius Springer, Berlin.

Fig. 1.

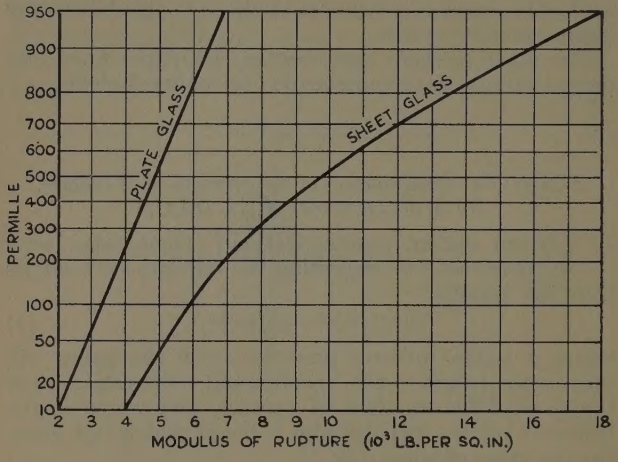

950
900
800
700
600
500
400
300
200
100
50
20
10
PERMILLE
PLATE GLASS
SHEET GLASS
2 3 4 5 6 7 8 9 10 12 14 16 18
MODULUS OF RUPTURE (10³ LB. PER SQ. IN.)

Fig. 2.

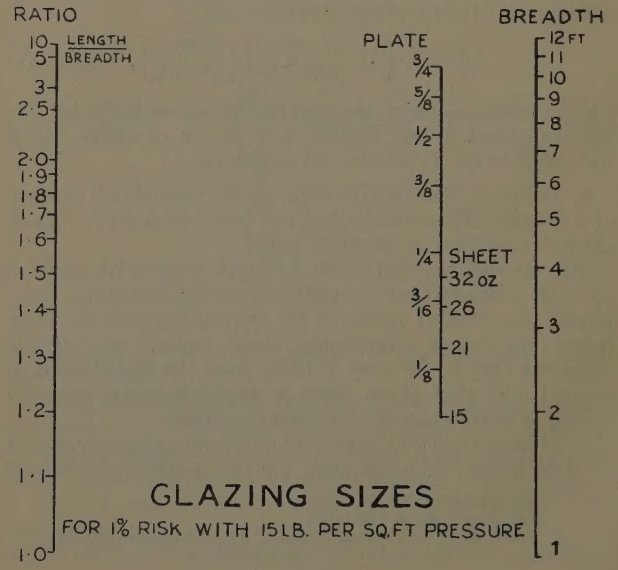

RATIO
LENGTH / BREADTH
10
5
3
2·5
2·0
1·9
1·8
1·7
1·6
1·5
1·4
1·3
1·2
1·1
1·0
PLATE
3/4
5/8
1/2
3/8
1/4
3/16
1/8
SHEET
32 oz
26
21
15
BREADTH
12 FT
11
10
9
8
7
6
5
4
3
2
1
GLAZING SIZES
FOR 1% RISK WITH 15 LB. PER SQ. FT PRESSURE

plate glass, and 4000 lb./sq. in. for the sheet glass, the risk of failure in each case is one per cent.

4. The nomogram shown in fig. 2 has been drawn on this basis for a pressure of 15 lb./sq. ft. *. The graduations of the scale for plate glass are the normal sizes of commerce, and for sheet glass the thicknesses are denoted, as is customary, by the number of ounces per sq. ft.

A straight line drawn across the chart from ratio 2 to breadth 3 ft. shows that for a pane of glass 6 ft. × 3 ft. neither 3/16-in. plate nor 26-oz. sheet is fully adequate for the assumed risk.

Building Research Station,
Garston, Herts.

C. *Notices respecting New Books.*

Reactions of Organic Compounds. By W. J. Hickinbottom, D.Sc., F.I.C. (London: Longmans, Green & Co., 1936. Price 16*s.* net.)

IN this book of over four hundred pages the author presents a very comprehensive account of the reactions of the principal classes of organic compounds, one or more chapters being devoted to each main type. The principal synthetic methods of organic chemistry are described under the appropriate class of compound, and particular care has been taken to indicate the limits within which these reactions may be applied, particular emphasis being laid on cases where well-known reactions proceed abnormally. This is exemplified by the treatment of the Friedel-Crafts reaction, the bromination and also the thermal decomposition of dialkylanilines and the halogenation of substituted anilines and sulphanilic acids. The apparent abnormalities in the behaviour of various groups in the benzene nucleus under the influence of other substituents receive attention, and the author is thoroughly alive to the importance of recent work on the mechanism of reaction. Thus, reference is made to investigations on the addition of halogen acids to olefines, to the mechanism of exhaustive methylation, and also to quantitative experiments on the relative proportions of nitrite and nitro-compound formed from alkyl halides and silver nitrite.

The additive powers of the radical thiocyanogen are men-

* *Cf.* L.C.C. 'Code of Practice,' 1932.

tioned, although the equally interesting cases where it causes substitution in aromatic and heterocyclic nuclei are omitted.

Apart from references to the interaction of unsaturated hydrocarbons with mercury salts, little is mentioned concerning mercuration. This might possibly be considered in a future edition along with some account of the properties of the thiophen nucleus. These omissions, however, in no way detract from the merits of the work. Much of the subject-matter is not to be found elsewhere except in the original literature or in very much larger works. Details for the preparation of a large number of compounds are given, and numerous references appear on almost every page. The index is extensive.

The book will be useful to the Honours student, and invaluable to the research worker whether in his early apprenticeship or in his later years. An Appendix of 21 pages is devoted to the development of a scheme for the identification of organic compounds of fairly simple structure. The paragraphs in smaller type contain very useful hints, and the undergraduate who is already fairly well-equipped will benefit. Opinions may, however, differ as to the necessity for this Appendix, and its deletion from a subsequent edition might enable the author to extend his excellent method of treatment to a few compounds and reactions at present omitted.

A Course of Integrational and Operational Calculus: with Applications to Problems of Physics and Electrotechnics. By Konstantin Dahr. [8vo, pp. 169, 12 figs. Unbound.] (Stockholm: Lindstahl.)

This treatment of the operational methods introduced by Heaviside is based from the outset on the Fourier-Mellin integral theorem, a proof of which is given in an appendix. Thus it is in the direct tradition of Bromwich's paper of 1916. More attention is given to mathematical rigour than in other expositions known to me; in particular the decomposition of such operators as cosech ph into infinite series of partial fractions, and their interpretation by the partial-fraction rule, are justified by a general theorem and not as individual cases. Bessel functions and some asymptotic expansions are treated. References in the text, however, are scanty, and though there is a useful bibliography at the end it gives no papers before 1923; thus Bromwich, who revolutionized the mathematical attitude to the methods, and whose fundamental integral appears on p. 6, has escaped mention. Apart from this omission the treatment is excellent.

[*The Editors do not hold themselves responsible for the views expressed by their correspondents.*]